Core Logic as the Point of Convergence of Different Lines of Investigation

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The Main Aim

Devise a system of formal logic that will enable one to regiment deductive reasoning in mathematics.

This was Frege's main aim (for classical mathematics).

It has been achieved only imperfectly.

One can pursue the same aim for constructive mathematics.

Let us consider very carefully exactly what is required of any proof system for it to be adequate unto the demands of formalization (regimentation) of deductive reasoning in mathematics and scientific theory testing.

Desiderata for a Logical System

A logical system $\mathcal{S}$—say, for definiteness, a system of natural deduction—that satisfies the following five conditions is adequate for all the deductive demands of mathematics and science.

1. $\mathcal{S}$-proofs are finite. Hence, the set of premises of any $\mathcal{S}$-proof is finite. The relation 'Π is an $\mathcal{S}$-proof of the sentence φ whose premises form the set Δ' is effectively decidable.
2. If there is an $\mathcal{S}$-proof of $\bot$ whose premises form the set Δ, then Δ has no model.
3. If Δ is a set of sentences that has no model, then there is an $\mathcal{S}$-proof of absurdity ($\bot$) from premises in Δ.
4. If Δ has a model and there is an $\mathcal{S}$-proof of φ from premises in Δ, then every model of Δ makes φ true.
5. If Δ has a model and every model of Δ makes φ true, then there is an $\mathcal{S}$-proof of φ from premises in Δ.

Two extra conditions on a proof system

If the logical system $\mathcal{S}$ satisfies in addition the following two extra conditions, then it is *superbly* adequate for all the deductive demands of mathematics and science.

6. The premise set Δ and the conclusion φ of any $\mathcal{S}$-proof are mutually relevant.
7. Given any *informal* proof P in a mathematics journal or textbook, written by an expert mathematician, and acknowledged within the pertinent mathematical community as meeting their standards of informal rigor, the following task will be feasible for competent users of system $\mathcal{S}$: extract from the proof P its conclusion φ, along with its set Δ of premises; then regiment P as a fully formal $\mathcal{S}$-proof of φ from Δ, in such a way as to merely supply missing detail, while preserving at the macro-level all the various 'lines of argument' that an expert mathematician can discern within P.

The methodological adequacy of the Core systems

Core Logic $\mathbb{C}$ satisfies conditions (1)-(7) for intuitionistic mathematics (and for empirical theory-testing).

Classical Core Logic $\mathbb{C}^+$ satisfies conditions (1)-(7) for classical mathematics.

The different lines of investigation that lead to $\mathbb{C}$ and $\mathbb{C}^+$

1. **Relevantize.** This means, above all else: *avoid the First Lewis Paradox*. Relevantize *deducibility* ($\vdash$), not the conditional ($\rightarrow$). Relevantize both constructive and classical reasoning *in the same way*. In natural deduction, *have all proofs in normal form*, since abnormality is a source of irrelevance. In sequent calculus, *avoid structural rules* (except for Reflexivity), since Cut and Thinning are sources of irrelevance.
2. **Respect the truth tables.** Show how the truth tables *are the inferential source of connective-meanings*; and *are the source of deductive rules of inference*.
3. **Pursue epistemic gain.** Prefer a proof of any proper subsequent of $\Delta : \varphi$ to a proof of $\Delta : \varphi$ itself; this is *epistemic gain*. Devise rules of inference that afford proof-search methods that exploit epistemic gains.
4. **Take seriously the idea that proofs are objects of search.** Devise rules of inference that afford well-constrained search for proofs. To this end, exploit any potential for epistemic gain, and exploit relevance of premises to conclusions. Also, make natural deductions essentially isomorphic to their corresponding sequent proofs.

The different lines of investigation that lead to $\mathbb{C}$ and $\mathbb{C}^+$

5. **Treat only single-conclusion logic.** Do not be concerned with how often a premise may be used in a proof. In the case of the sequent calculus, deal only with *set* sequents whose succedents are at most singletons.
6. **Address afresh the role of absurdity ($\perp$) and the problem of how to handle inconsistency.** Be afraid of inconsistency. Be very afraid. But not because of Explosion!—absurdity, on its own, is 'bad enough'.
7. **Accommodate transitivity of deduction in mathematics as we find it**. Respect, and take advantage of, the mathematicians' practice of interpolating judiciously chosen lemmas to reduce deductive load. Deliberate *au fond* about what is *needed* of a system of proof that is to regiment deduction faithfully in actual mathematical practice. Be prepared to challenge some deeply entrenched metalogical dogmas in the current tradition.
8. [Not for today's talk:] Provide the reasoning needed to reveal paradox.
9. [Not for today's talk:] Provide the meta-reasoning needed for rational belief revision.

Let us see now how to avoid Lewis's First paradox, which is anathema to the relevantist.

Lewis's First Paradox

Positive form:

$$A, \neg A : B$$

(in **I**, hence also in **C**)

Negative form:

$$A, \neg A : \neg B$$

(in **M**, hence also in **I** and **C**)

Sources of the *inferential form* of Lewis's First Paradox

Sources of the positive form $A, \neg A : B$

EFQ (in ND); vacuous discharge with CR (in ND); Thinning (in SC); Cut (in SC)

$$\dfrac{\dfrac{A \quad \neg A}{\bot}}{B}\,\text{EFQ} \qquad \dfrac{\dfrac{A \quad \neg A}{\bot}}{B}\,\text{CR} \qquad \dfrac{\dfrac{A : A}{A, \neg A :}}{A, \neg A : B}\,\text{Thinning} \qquad \dfrac{\dfrac{A : A}{A : A \vee B} \qquad \dfrac{\dfrac{A : A}{A, \neg A :} \qquad B : B}{A \vee B, \neg A : B}}{A, \neg A : B}\,\text{Cut}$$

Sources of the negative form $A, \neg A : \neg B$

EFQ (in ND); vacuous discharge with $\neg$-I (in ND); Thinning (in SC); Cut (in SC)

$$\dfrac{\dfrac{A \quad \neg A}{\bot}}{\neg B}\,\text{EFQ} \qquad \dfrac{\dfrac{A \quad \neg A}{\bot}}{\neg B}\,\neg\text{-I} \qquad \dfrac{\dfrac{A : A}{A, \neg A :}}{A, \neg A : \neg B}\,\text{Thinning} \qquad \dfrac{\dfrac{A : A}{A : A \vee \neg B} \qquad \dfrac{\dfrac{A : A}{A, \neg A :} \qquad \neg B : \neg B}{A \vee \neg B, \neg A : \neg B}}{A, \neg A : \neg B}\,\text{Cut}$$

Ways of making premises spuriously relevant to conclusions.

In ND:

$$\dfrac{\dfrac{A \qquad B}{A \wedge B}\,\neg\text{-I}}{B}\,\neg\text{-E}$$
$$\vdots$$
$$C$$

Premise A has been made spuriously relevant to conclusion C.

The culprit is the maximal occurrence of $A \wedge B$.

The proof is not in normal form.

In SC:

$$\dfrac{\dfrac{A : A \quad B : B}{A, B : A \wedge B} \qquad \dfrac{\begin{matrix}\vdots \\ B : C\end{matrix}}{A \wedge B : C}}{A, B : C}\,\text{Cut}$$

Again, premise A has been made spuriously relevant to conclusion C.

The proof is not cut-free.

We see that in order to avoid the First Lewis Paradox, natural deductions will have to be in normal form; sequent proofs will have to be cut-free and thinning-free; and vacuous discharge of assumptions will have to be prohibited in application of the rules of Classical Reductio and $\neg$-Introduction.

Let me introduce you now to an inferentialist construal of the familiar truth tables.

Model-relative *evaluation* rules will be formulated. These are rules of *verification* and rules of *falsification*. They generate verifications, or *truthmakers*, and falsifications, or *falsitymakers*, of sentences with respect to a given model. They explicate the ways in which sentences acquire their truth values in a given model.

Note how *relevant* all the evaluative inferences will be.

Rules of Verification and Falsification for ¬

(¬-$\mathcal{V}$)

$$\dfrac{\begin{array}{c}\underbrace{\overset{\Box\!\!-\!\!-\!(i)}{\varphi}\ ,\ \Lambda}\\ \vdots\\ \bot\end{array}}{\neg\varphi}\,(i)$$

To verify a negation $\neg\varphi$ one must falsify φ.

(¬-$\mathcal{F}$)

$$\dfrac{\neg\varphi \qquad \begin{array}{c}\Lambda\\ \vdots\\ \varphi\end{array}}{\bot}$$

To falsify a negation $\neg\varphi$ one must verify φ.

Note the classicism that has crept in by virtue of this dualizing.

Rules of Verification and Falsification for $\wedge$

$(\wedge\text{-}\mathcal{V})$

$$\dfrac{\begin{array}{c}\Lambda_1\\ \vdots\\ \varphi\end{array}\qquad\begin{array}{c}\Lambda_2\\ \vdots\\ \psi\end{array}}{\varphi \wedge \psi}$$

To verify a conjunction, one must verify both conjuncts.

$(\wedge\text{-}\mathcal{F})$

$$\dfrac{\varphi \wedge \psi \qquad \begin{array}{c}\overset{\square\text{—}(i)}{\underbrace{\varphi\ ,\ \Lambda}}\\ \vdots\\ \bot\end{array}}{\bot}\,(i) \qquad\qquad \dfrac{\varphi \wedge \psi \qquad \begin{array}{c}\overset{\square\text{—}(i)}{\underbrace{\psi\ ,\ \Lambda}}\\ \vdots\\ \bot\end{array}}{\bot}\,(i)$$

To falsify a conjunction, one must falsify one of the conjuncts.

Rules of Verification and Falsification for $\vee$

$(\vee\text{-}\mathcal{V})$

$$\begin{array}{cc} \Lambda & \Lambda \\ \vdots & \vdots \\ \dfrac{\varphi}{\varphi \vee \psi} & \dfrac{\psi}{\varphi \vee \psi} \end{array}$$

To verify a disjunction, one must verify one of the disjuncts.

$(\vee\text{-}\mathcal{F})$

$$\dfrac{\varphi \vee \psi \qquad \begin{array}{c} \underbrace{\overset{\square\!-\!(i)}{\varphi}\ ,\ \Lambda_1} \\ \vdots \\ \bot \end{array} \qquad \begin{array}{c} \underbrace{\overset{\square\!-\!(i)}{\psi}\ ,\ \Lambda_2} \\ \vdots \\ \bot \end{array}}{\bot}\,(i)$$

To falsify a disjunction, one must falsify both disjuncts.

Rules of Verification and Falsification for $\rightarrow$

$(\rightarrow\text{-}\mathcal{V})$

$$\dfrac{\begin{array}{c}\overset{\square\text{---}(i)}{\underbrace{\varphi\ ,\ \Lambda}}\\ \vdots \\ \bot\end{array}}{\varphi \rightarrow \psi}\,(i) \qquad \dfrac{\begin{array}{c}\Lambda\\ \vdots \\ \psi\end{array}}{\varphi \rightarrow \psi}$$

To verify a conditional, one must either falsify its antecedent or verify its consequent.

$(\rightarrow\text{-}\mathcal{F})$

$$\dfrac{\varphi \rightarrow \psi \qquad \begin{array}{c}\Lambda_1\\ \vdots \\ \varphi\end{array} \qquad \begin{array}{c}\overset{\square\text{---}(i)}{\underbrace{\psi\ ,\ \Lambda_2}}\\ \vdots \\ \bot\end{array}}{\bot}\,(i)$$

To falsify a conditional, one must both verify its antecedent and falsify its consequent.

Rules of Verification and Falsification for ∃

$$(\exists\text{-}\mathcal{V})\qquad \dfrac{\begin{matrix}\Lambda\\ \vdots\\ \varphi^x_{\alpha}\end{matrix}}{\exists x\varphi}$$

where α is any individual in the domain

To verify an existential, one must verify some instance of it.

$$(\exists\text{-}\mathcal{F})\qquad \dfrac{\exists x\varphi \quad \begin{matrix}\overline{\Box}^{(i)}\\ \underbrace{\varphi^x_{\alpha_1}\ ,\ \Lambda_1}\\ \vdots\\ \bot\end{matrix} \quad \dots \quad \begin{matrix}\overline{\Box}^{(i)}\\ \underbrace{\varphi^x_{\alpha_n}\ ,\ \Lambda_n}\\ \vdots\\ \bot\end{matrix} \quad \dots}{\bot}\,(i)\ M$$

where $\alpha_1, \dots, \alpha_n, \dots$ are all the individuals in the domain of M

To falsify an existential, one must falsify every instance of it.

Rules of Verification and Falsification for ∀

$$(\forall\text{-}\mathcal{V}) \qquad \dfrac{\begin{array}{cccc}\Lambda_1 & & \Lambda_n & \\ \vdots & \ldots & \vdots & \ldots \\ \psi(\boldsymbol{\alpha_1}) & & \psi(\boldsymbol{\alpha_n}) & \end{array}}{\forall x\psi(x)}\,M$$

where $\boldsymbol{\alpha_1},\ldots,\boldsymbol{\alpha_n},\ldots$ are all the individuals in the domain of M

To verify a universal, one must verify every instance of it.

$$(\forall\text{-}\mathcal{F}) \qquad \dfrac{\forall x\psi(x) \qquad \begin{array}{c} \underbrace{\overset{\square\overline{\quad\quad}(i)}{\psi(\boldsymbol{\alpha})}\ ,\ \ \Lambda} \\ \vdots \\ \bot \end{array}}{\bot}\,(i)$$

where $\boldsymbol{\alpha}$ is any individual in the domain of M

To falsify a universal, one must falsify some instance of it.

Let us see now how these model-*relative* evaluation rules are the source of model-*invariant* rules of natural deduction.

Morphing Model-Relative Verification Rules into Model-Invariant Introduction Rules: $\neg$-$\mathcal{V}$ to $\neg$-I

$$(\neg\text{-}\mathcal{V}) \quad \dfrac{\begin{matrix} \overset{\square\!\!-\!\!-(i)}{\underbrace{\varphi \;\; , \;\; \Lambda}} \\ F \\ \bot \end{matrix}}{\neg\varphi}(i) \qquad \rightsquigarrow \qquad (\neg\text{-I}) \quad \dfrac{\begin{matrix} \overset{\square\!\!-\!\!-(i)}{\underbrace{\varphi \;\; , \;\; \Delta}} \\ \Pi \\ \bot \end{matrix}}{\neg\varphi}(i)$$

Morphing Model-Relative Verification Rules into Model-Invariant Introduction Rules: $\wedge$-$\mathcal{V}$ to $\wedge$-I

$$(\wedge\text{-}\mathcal{V}) \quad \dfrac{\begin{array}{cc} \Lambda_1 & \Lambda_2 \\ V_1 & V_2 \\ \varphi_1 & \varphi_2 \end{array}}{\varphi_1 \wedge \varphi_2} \qquad \rightsquigarrow \qquad (\wedge\text{-I}) \quad \dfrac{\begin{array}{cc} \Delta_1 & \Delta_2 \\ \Pi_1 & \Pi_2 \\ \varphi_1 & \varphi_2 \end{array}}{\varphi_1 \wedge \varphi_2}$$

Morphing Model-Relative Verification Rules into Model-Invariant Introduction Rules: $\vee$-$\mathcal{V}$ to $\vee$-I

$$(\vee\text{-}\mathcal{V}) \quad \dfrac{\begin{array}{c}\Lambda \\ V \\ \varphi\end{array}}{\varphi \vee \psi} \quad \dfrac{\begin{array}{c}\Lambda \\ V \\ \psi\end{array}}{\varphi \vee \psi} \qquad \rightsquigarrow \qquad (\vee\text{-I}) \quad \dfrac{\begin{array}{c}\Delta \\ \Pi \\ \varphi\end{array}}{\varphi \vee \psi} \quad \dfrac{\begin{array}{c}\Delta \\ \Pi \\ \psi\end{array}}{\varphi \vee \psi}$$

Morphing Model-Relative Verification Rules into Model-Invariant Introduction Rules: $\to$-$\mathcal{V}$ to $\to$-I

$$(\to\text{-}\mathcal{V})\qquad \frac{\begin{array}{c}\underbrace{\overset{\square\!-\!(i)}{\varphi}\ ,\ \Lambda}_{F}\\ \bot\end{array}}{\varphi\to\psi}\,(i)\qquad \frac{\begin{array}{c}\Lambda\\ V\\ \psi\end{array}}{\varphi\to\psi}\qquad\rightsquigarrow\qquad(\to\text{-I})\qquad \frac{\begin{array}{c}\underbrace{\overset{\square\!-\!(i)}{\varphi}\ ,\ \Delta}_{\Pi}\\ \bot\end{array}}{\varphi\to\psi}\,(i)\qquad \frac{\begin{array}{c}\Delta\\ \Pi\\ \psi\end{array}}{\varphi\to\psi}$$

Note that these two parts of ($\to$-I) resulting from this direct morphing of the verification rule ($\to$-$\mathcal{V}$) do not yet furnish that form of ($\to$-I) that permits one to assume φ for the sake of argument, then to deduce ψ by using assumption φ, and finally to discharge φ when one infers the conclusion $\varphi\to\psi$. We shall see in due course how to get this 'missing part' of ($\to$-I).

Morphing Model-Relative Verification Rules into Model-Invariant Introduction Rules: $\forall$-$\mathcal{V}$ to $\forall$-I

$$(\forall\text{-}\mathcal{V})\qquad \frac{\left\{\begin{matrix}\Lambda_{\boldsymbol{\alpha}}\\ V_{\boldsymbol{\alpha}}\\ \psi(\boldsymbol{\alpha})\end{matrix}\right\}_{\boldsymbol{\alpha}\in M}}{\forall x\psi(x)} \qquad \rightsquigarrow \qquad (\forall\text{-I})\qquad \frac{\begin{matrix}\Delta^{\not a}\\ \Pi\\ \psi(a)\end{matrix}}{\forall x\psi(x)}$$

Morphing Model-Relative Verification Rules into Model-Invariant Introduction Rules: $\exists$-$\mathcal{V}$ to $\exists$-I

$$(\exists\text{-}\mathcal{V}) \quad \frac{\begin{array}{c}\Lambda \\ V \\ \varphi^x_{\boldsymbol{\alpha}}\end{array}}{\exists x\varphi}$$ where $\boldsymbol{\alpha}$ is any individual in the domain

$\rightsquigarrow$

$$(\exists\text{-I}) \quad \frac{\begin{array}{c}\Delta \\ \Pi \\ \varphi^x_t\end{array}}{\exists x\varphi}$$ where t is any closed term

Morphing Model-Relative Falsification Rules into Model-Invariant Elimination Rules: ¬-$\mathcal{F}$ to ¬-E

$$(\neg\text{-}\mathcal{F})\qquad \dfrac{\neg\varphi \qquad \begin{matrix}\Lambda\\ V\\ \varphi\end{matrix}}{\bot} \qquad \rightsquigarrow \qquad (\neg\text{-E})\qquad \dfrac{\neg\varphi \qquad \begin{matrix}\Delta\\ \Pi\\ \varphi\end{matrix}}{\bot}$$

Morphing Model-Relative Falsification Rules into Model-Invariant Elimination Rules: $\wedge$-$\mathcal{F}$ to $\wedge$-E

$$(\wedge\text{-}\mathcal{F}) \quad \dfrac{\varphi\wedge\psi \qquad \underset{\bot}{\underbrace{\overset{\square\overline{\quad}(i)}{\varphi} \;,\; \Lambda}_{F}}}{\bot}\,(i) \qquad \dfrac{\varphi\wedge\psi \qquad \underset{\bot}{\underbrace{\overset{\square\overline{\quad}(i)}{\psi} \;,\; \Lambda}_{F}}}{\bot}\,(i) \qquad \rightsquigarrow$$

$$(\wedge\text{-E}) \quad \dfrac{\varphi\wedge\psi \qquad \underset{\theta/\bot}{\underbrace{\overset{\square\overline{\quad}(i)}{\varphi} \;,\; \Delta}_{\Pi}}}{\theta/\bot}\,(i) \qquad \dfrac{\varphi\wedge\psi \qquad \underset{\theta/\bot}{\underbrace{\overset{\square\overline{\quad}(i)}{\psi} \;,\; \Delta}_{\Pi}}}{\theta/\bot}\,(i) \;;\; \text{hence, more efficiently:} \quad \dfrac{\varphi\wedge\psi \qquad \underset{\theta/\bot}{\underbrace{\overset{(i)\overline{\quad}\square\overline{\quad}(i)}{\varphi\,,\,\psi} \;,\; \Delta}_{\Pi}}}{\theta/\bot}\,(i) \;,$$

where the box between the two discharge strokes indicates that at least one of φ and ψ must feature as an undischarged assumption of the subproof Π.

Morphing Model-Relative Falsification Rules into Model-Invariant Elimination Rules: $\vee$-$\mathcal{F}$ to $\vee$-E

$$
(\vee\text{-}\mathcal{F})\quad \dfrac{\varphi_1 \vee \varphi_2 \qquad \begin{matrix} \overset{\square\!-\!(i)}{\underbrace{\varphi_1 \;,\; \Lambda_1}} \\ F_1 \\ \bot \end{matrix} \qquad \begin{matrix} \overset{\square\!-\!(i)}{\underbrace{\varphi_2 \;,\; \Lambda_2}} \\ F_2 \\ \bot \end{matrix}}{\bot}\,(i)
$$

$$
\rightsquigarrow \qquad (\vee\text{-E})\quad \dfrac{\varphi_1 \vee \varphi_2 \qquad \begin{matrix} \overset{\square\!-\!(i)}{\underbrace{\varphi_1 \;,\; \Delta_1}} \\ \Pi_1 \\ \theta/\bot \end{matrix} \qquad \begin{matrix} \overset{\square\!-\!(i)}{\underbrace{\varphi_2 \;,\; \Delta_2}} \\ \Pi_2 \\ \theta/\bot \end{matrix}}{\theta/\bot}\,(i)
$$

'If either of the two case-proofs Π_1, Π_2 has $\bot$ as its conclusion, bring down as the main conclusion the conclusion of the other case-proof.'

Morphing Model-Relative Falsification Rules into Model-Invariant Elimination Rules: $\to$-$\mathcal{F}$ to $\to$-E

$$(\to\text{-}\mathcal{F})\qquad \dfrac{\varphi\to\psi \qquad \begin{array}{c}\Lambda_1\\ V\\ \varphi\end{array} \qquad \begin{array}{c}\overset{\square\!-\!(i)}{\underbrace{\psi\ ,\ \Lambda_2}}\\ F\\ \bot\end{array}}{\bot}\,(i) \qquad\leadsto\qquad (\to\text{-E})\qquad \dfrac{\varphi\to\psi \qquad \begin{array}{c}\Delta_1\\ \Pi_1\\ \varphi\end{array} \qquad \begin{array}{c}\overset{\square\!-\!(i)}{\underbrace{\psi\ ,\ \Delta_2}}\\ \Pi_2\\ \theta/\bot\end{array}}{\theta/\bot}\,(i)$$

Morphing Model-Relative Falsification Rules into Model-Invariant Elimination Rules: $\forall$-$\mathcal{F}$ to $\forall$-E

$$(\forall\text{-}\mathcal{F})\qquad \dfrac{\forall x\psi(x) \qquad \begin{array}{c} \square\overline{\quad\quad}^{(i)} \\ \underbrace{\psi(\alpha)\ ,\ \Lambda}_{} \\ F \\ \bot \end{array}}{\bot}{\scriptstyle (i)}$$

where α is any individual in the domain

$\rightsquigarrow$

$$(\forall\text{-E})\qquad \dfrac{\forall x\psi(x) \qquad \begin{array}{c} \square\overline{\quad\quad}^{(i)} \\ \underbrace{\psi(t)\ ,\ \Delta}_{} \\ \Pi \\ \theta/\bot \end{array}}{\theta/\bot}{\scriptstyle (i)} \quad ; \quad \text{hence} \quad \dfrac{\forall x\psi(x) \qquad \begin{array}{c} {}^{(i)}\overline{\quad\quad}\ \cdots\ \square\ \cdots\ \overline{\quad\quad}^{(i)} \\ \underbrace{\psi(t_1)\ ,\ \ldots\ ,\ \psi(t_n)\ ,\ \Delta}_{} \\ \Pi \\ \theta/\bot \end{array}}{\theta/\bot}{\scriptstyle (i)}$$

where t is any closed term

where $t_1, \ldots, t_n$ are closed terms

Morphing Model-Relative Falsification Rules into Model-Invariant Elimination Rules: $\exists$-$\mathcal{F}$ to $\exists$-E

$$(\exists\text{-}\mathcal{F}) \quad \frac{\exists x\varphi \quad \left\{ \begin{matrix} \square\overline{\quad}^{(i)} \\ \underbrace{\varphi^x_\alpha \, , \, \Lambda_\alpha} \\ F_\alpha \\ \bot \end{matrix} \right\}_{\alpha \in M}}{\bot}{(i)} \qquad \rightsquigarrow \qquad (\exists\text{-E}) \quad \frac{\exists x\varphi^{\not a} \quad \begin{matrix} \square\overline{\quad}^{(i)} \\ \underbrace{\varphi^x_a \, , \, \Delta^{\not a}} \\ \Pi \\ \theta^{\not a}/\bot \end{matrix}}{\theta^{\not a}/\bot}{(i)}$$

Let us see now how morphing the model-relative Verification and Falsification rules produces not only the Introduction and Elimination rules (respectively) of Natural Deduction but also the directly corresponding Right- and Left-rules of the Sequent Calculus.

$(\neg\text{-}\mathcal{V})$

$$\dfrac{\begin{array}{c}\overset{\square\text{---}(i)}{\underbrace{\varphi\ ,\ \Lambda}}\\ \vdots \\ \bot\end{array}}{\neg\varphi}\,(i)$$

$(\neg\text{-I})$

$$\dfrac{\begin{array}{c}\overset{\square\text{---}(i)}{\underbrace{\varphi\ ,\ \Delta}}\\ \vdots \\ \bot\end{array}}{\neg\varphi}\,(i)$$

$(\neg\text{-R})$

$$\dfrac{\varphi,\ \Delta : \bot}{\Delta : \neg\varphi}$$

$(\neg\text{-}\mathcal{F})$

$$\dfrac{\neg\varphi \qquad \begin{array}{c}\Lambda\\ \vdots\\ \varphi\end{array}}{\bot}$$

$(\neg\text{-E})$

$$\dfrac{\neg\varphi \qquad \begin{array}{c}\Delta\\ \vdots\\ \varphi\end{array}}{\bot}$$

$(\neg\text{-L})$

$$\dfrac{\Delta : \varphi}{\Delta, \neg\varphi : \bot}$$

$$(\wedge\text{-}\mathcal{V}) \quad \dfrac{\begin{array}{cc}\Lambda_1 & \Lambda_2 \\ \vdots & \vdots \\ \varphi & \psi\end{array}}{\varphi \wedge \psi}$$

$$(\wedge\text{-I}) \quad \dfrac{\begin{array}{cc}\Delta_1 & \Delta_2 \\ \vdots & \vdots \\ \varphi & \psi\end{array}}{\varphi \wedge \psi}$$

$$(\wedge\text{-R}) \quad \dfrac{\Delta_1 : \varphi \qquad \Delta_2 : \psi}{\Delta, \Delta_2 : \varphi \wedge \psi}$$

$$(\wedge\text{-}\mathcal{F}) \quad \left\{ \begin{array}{l} \dfrac{\varphi \wedge \psi \qquad \begin{array}{c} \overset{\square\text{——}(i)}{\underbrace{\varphi \;,\; \Lambda}} \\ \vdots \\ \bot \end{array}}{\bot}(i) \\[3em] \dfrac{\varphi \wedge \psi \qquad \begin{array}{c} \overset{\square\text{——}(i)}{\underbrace{\psi \;,\; \Lambda}} \\ \vdots \\ \bot \end{array}}{\bot}(i) \end{array} \right.$$

$$(\wedge\text{-E}) \quad \dfrac{\varphi \wedge \psi \qquad \begin{array}{c} \overset{(i)\text{—}\square\text{—}(i)}{\underbrace{\varphi , \psi \;,\; \Delta}} \\ \vdots \\ \theta \end{array}}{\theta}(i)$$

$$(\wedge\text{-L}) \quad \left\{ \begin{array}{c} \dfrac{\Delta, \varphi, \psi : \theta}{\Delta, \varphi \wedge \psi : \theta} \\[1em] \dfrac{\Delta, \varphi : \theta}{\Delta, \varphi \wedge \psi : \theta} \\[1em] \dfrac{\Delta, \psi : \theta}{\Delta, \varphi \wedge \psi : \theta} \end{array} \right.$$

(∨-$\mathcal{V}$)

$$\dfrac{\begin{array}{c}\Lambda\\ \vdots\\ \varphi\end{array}}{\varphi \vee \psi} \qquad \dfrac{\begin{array}{c}\Lambda\\ \vdots\\ \psi\end{array}}{\varphi \vee \psi}$$

(∨-I)

$$\dfrac{\begin{array}{c}\Delta\\ \vdots\\ \varphi\end{array}}{\varphi \vee \psi} \qquad \dfrac{\begin{array}{c}\Delta\\ \vdots\\ \psi\end{array}}{\varphi \vee \psi}$$

(∨-R)

$$\dfrac{\Delta : \varphi}{\Delta : \varphi \vee \psi} \qquad \dfrac{\Delta : \psi}{\Delta : \varphi \vee \psi}$$

(∨-$\mathcal{F}$)

$$\dfrac{\varphi \vee \psi \qquad \begin{array}{c}\overline{\Box}^{(i)}\\ \underbrace{\varphi\ ,\Lambda_1}\\ \vdots\\ \bot\end{array} \qquad \begin{array}{c}\overline{\Box}^{(i)}\\ \underbrace{\psi\ ,\Lambda_2}\\ \vdots\\ \bot\end{array}}{\bot}\,(i)$$

(∨-E)

$$\dfrac{\varphi \vee \psi \qquad \begin{array}{c}\overline{\Box}^{(i)}\\ \underbrace{\varphi\ ,\Delta_1}\\ \vdots\\ \theta/\bot\end{array} \qquad \begin{array}{c}\overline{\Box}^{(i)}\\ \underbrace{\psi\ ,\Delta_2}\\ \vdots\\ \theta/\bot\end{array}}{\theta/\bot}\,(i)$$

(∨-L)

$$\dfrac{\Delta_1, \varphi : \theta/\bot \qquad \Delta_2, \psi : \theta/\bot}{\Delta_1, \Delta_2, \varphi \vee \psi : \theta/\bot}$$

$$(\rightarrow\text{-}\mathcal{V})\quad \dfrac{\begin{array}{c}\overset{\square\text{---}(i)}{\underbrace{\varphi\;,\;\Lambda}}\\ \vdots\\ \bot\end{array}}{\varphi\rightarrow\psi}\,(i)\qquad \dfrac{\begin{array}{c}\Lambda\\ \vdots\\ \psi\end{array}}{\varphi\rightarrow\psi}$$

$$(\rightarrow\text{-I})\quad \dfrac{\begin{array}{c}\overset{\square\text{---}(i)}{\underbrace{\varphi\;,\;\Delta}}\\ \vdots\\ \bot\end{array}}{\varphi\rightarrow\psi}\,(i)\qquad \dfrac{\begin{array}{c}\overset{\diamond\text{---}(i)}{\underbrace{\varphi\;,\;\Delta}}\\ \vdots\\ \psi\end{array}}{\varphi\rightarrow\psi}\,(i)$$

$$(\rightarrow\text{-R})\quad \dfrac{\Delta,\varphi:\bot}{\Delta:\varphi\rightarrow\psi}\qquad \dfrac{\Delta:\psi}{\Delta\setminus\{\varphi\}:\varphi\rightarrow\psi}$$

$$(\rightarrow\text{-}\mathcal{F})\quad \dfrac{\varphi\rightarrow\psi\qquad \begin{array}{c}\Lambda_1\\ \vdots\\ \varphi\end{array}\qquad \begin{array}{c}\overset{\square\text{---}(i)}{\underbrace{\psi\;,\Lambda_2}}\\ \vdots\\ \bot\end{array}}{\bot}\,(i)$$

$$(\rightarrow\text{-E})\quad \dfrac{\varphi\rightarrow\psi\qquad \begin{array}{c}\Delta_1\\ \vdots\\ \varphi\end{array}\qquad \begin{array}{c}\overset{\square\text{---}(i)}{\underbrace{\psi\;,\Delta_2}}\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

$$(\rightarrow\text{-L})\quad \dfrac{\Delta_1:\varphi\qquad \Delta_2,\psi:\theta}{\Delta_1,\Delta_2,\varphi\rightarrow\psi:\theta}$$

(∃-$\mathcal{V}$)

$$\begin{array}{c}\Lambda\\ \vdots\\ \underline{\varphi^x_\alpha}\\ \exists x\varphi\end{array}$$

where $\alpha \in M$

(∃-I)

$$\begin{array}{c}\Delta\\ \vdots\\ \underline{\varphi^x_t}\\ \exists x\varphi\end{array}$$

(∃-R)

$$\frac{\Delta : \varphi^x_t}{\Delta : \exists x\varphi}$$

(∃-$\mathcal{F}$)

$$\frac{\exists x\varphi \quad \left\{\begin{array}{c}\underbrace{\overline{\varphi^x_\alpha}^{\,\square\,(i)}\ ,\ \Lambda_\alpha}\\ \vdots\\ \bot\end{array}\right\}^{\alpha\in M}}{\bot}\,(i)$$

(∃-E)

$$\frac{\exists x\varphi^{ⓐ} \quad \begin{array}{c}\underbrace{\overline{\varphi^x_a}^{\,\square\,(i)}\ ,\ \Delta^{ⓐ}}\\ \vdots\\ \theta^{ⓐ}\end{array}}{\theta^{ⓐ}}\,(i)$$

(∃-L)

$$\frac{\Delta, \varphi^x_a : \theta}{\Delta, \exists x\varphi : \theta}$$

where a does not occur in the bottom sequent

(∀-$\mathcal{V}$)

$$\dfrac{\left\{\begin{array}{c}\Lambda_{\alpha}\\ \vdots\\ \psi(\alpha)\end{array}\right\}^{\alpha\in M}}{\forall x\psi(x)}$$

(∀-I)

$$\dfrac{\begin{array}{c}\Delta^{\textcircled{a}}\\ \vdots\\ \psi(a)\end{array}}{\forall x\psi(x)}$$

(∀-R)

$$\dfrac{\Delta : \psi(a)}{\Delta : \forall x\psi(x)}$$

where a does not occur in the bottom sequent

(∀-$\mathcal{F}$)

$$\dfrac{\forall x\psi \qquad \begin{array}{c}\overset{\square\overline{\quad}(i)}{\underbrace{\psi^{x}_{\alpha}\ ,\ \Lambda}}\\ \vdots\\ \bot\end{array}}{\bot}(i)$$

where $\alpha \in M$

(∀-E)

$$\dfrac{\forall x\psi \qquad \begin{array}{c}(i)\overline{\quad}\cdots\square\cdots\overline{\quad}(i)\\ \underbrace{\psi^{x}_{t_1},\ldots,\psi^{x}_{t_n}\ ,\ \Delta}\\ \vdots\\ \theta\end{array}}{\theta}(i)$$

where $t_1,\ldots,t_n$ are closed terms

(∀-L)

$$\dfrac{\Delta,\psi^{x}_{t_1},\ldots,\psi^{x}_{t_n} : \theta}{\Delta,\forall x\psi : \theta}$$

Graphic Rules for Core Logic

Some important reforms/innovations that result from morphing:

- Major premises for eliminations (MPEs) *stand proud*
- So, all core proofs are in *normal form*
- *Vacuous discharge* of assumptions is prohibited where need be
- The rules of $\rightarrow$-I and $\vee$-E are *liberalized*
- The rule Ex Falso Quodlibet is banned
- So, all proofs are *relevant*
- All core rules of inference (including those for Classical Core Logic) are subject to the Global Anti-Dilution Constraint on Formation of Proofs: a contemplated rule-application is not permitted if the sequent that would thereby be 'proved' dilutes (i.e. contains as a subsequent) any sequent established by a subproof of the immediate subproofs for the application of the rule in question. This is always an effectively decidable matter.

The different lines of investigation, cont.

Conceive of the truth of a sentence φ in a model *M* as consisting in the *existence* of an *M*-relative *truthmaker* for φ. Such a truthmaker will be a suitably structured *mathematical object* built up from ingredients of *M*, and components of φ. There are, in general, different ways that a sentence φ can be true in a model *M*. Each of these ways will be *reifiable* as a particular *M*-relative *truthmaker* for φ. This gives us a finer-grained analysis of truth and truth-conditions than we have from Tarski's recursive definitions of conditions of satisfaction and truth. We shall have

$$\varphi \text{ is true in } M \text{ in the sense of Tarski}$$

$$\Leftrightarrow$$

$$\exists\Pi(\Pi \text{ is an } M\text{-relative truthmaker for } \varphi)$$

In notation to be defined, this can be expressed as follows:

$$M \Vdash \varphi \Leftrightarrow \exists\Pi\, \mathcal{V}(\Pi, \varphi, M)$$

Equivalence with truth as defined by Tarski

Theorem

Modulo *a metatheory which contains the mathematics of* $\overline{\overline{D}}$*-furcating trees of finite depth, we have, for all models M with domain D,*

$$\exists\Pi\ \mathcal{V}(\Pi, \varphi, M, D) \quad \Leftrightarrow \quad M \Vdash \varphi, \text{ i.e., } \varphi \text{ is true in } M$$

where the right-hand side is in the sense of Tarski.

Tarskian truth (in a model M) *consists in* the *existence of an M-relative verification* of the sentence in question.

Generalizing an idea of Prawitz that dealt with the intuitionistic case, with inferential bases instead of models, and with basis-relative canonical proofs instead of model-relative verifications

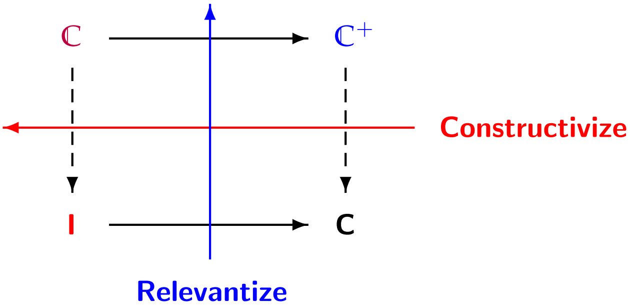
There is a 'quasi-effective' operation f such that for all models M:

$$
\left.\begin{array}{r l}
M\text{-relative verifications} & \left\{ \begin{array}{ccc} \Lambda_1 & & \Lambda_n \\ V_1 & ,\ldots, & V_n \\ \varphi_1 & & \varphi_n \end{array} \right. \\[2em]
\mathbb{C}^+\text{-proof} & \left\{ \begin{array}{c} \underbrace{\varphi_1 \;,\; \ldots \;,\varphi_n} \\ \Pi \\ \psi \end{array} \right.
\end{array}\right\}
\overset{f}{\longmapsto}
\left.\begin{array}{l} \Lambda \subseteq \sum_{i=1}^{n} \Lambda_i \\ V \\ \psi \end{array}\right\} M\text{-relative verification}
$$

Let us see now how Core Logic $\mathbb{C}$ and Classical Core Logic $\mathbb{C}^+$ relate to other main systems of logic with which we are all familiar.

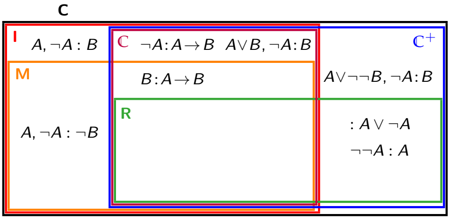
The right logic $\mathbb{C}$ for constructivist reasoning should bear to the right logic $\mathbb{C}^+$ for non-constructivist reasoning the same sort of relation (represented by the solid black horizontal arrows below) that Intuitionistic Logic **I** bears to Classical Logic **C**.

Likewise, the right logic $\mathbb{C}$ for constructivist reasoning should bear to Intuitionistic Logic **I** the same sort of relation (represented by the dashed black vertical arrows below) that the right logic $\mathbb{C}^+$ for non-constructivist reasoning bears to Classical Logic **C**.



Left-to-right is *classicizing*; top-to-bottom is *de-relevantizing*.

The Rival Systems



C : Classical (Frege, Russell; 1879)

I : Intuitionistic (Brouwer, Heyting; 1930)

M : Minimal (Johansson; 1936)

R : Relevant (Anderson–Belnap; 1962)

$\mathbb{C}$ **: Core**

$\mathbb{C}^{+}$ **: Classical Core**

$(\mathbf{I} \cap \mathbf{R}) \subset \mathbf{M} \subset \mathbf{I} \subset \mathbf{C}$

$(\mathbf{I} \cap \mathbf{R}) \subset \mathbb{C} = (\mathbb{C}^{+} \cap \mathbf{I}) \subset \mathbf{I} \subset \mathbf{C}$

$\mathbf{R} \subset \mathbb{C}^{+} \subset \mathbf{C}$

C is the ‘union’ of $\mathbb{C}^{+}$ **and I**

(The thin horizontal white zones are empty.)

Let us see now how the three standard systems **C**, **I**, and **M** contain irrelevancies, because of inattention to the effects of allowing vacuous discharge of assumptions, and insufficient flexibility in formulating the rules $\rightarrow$-I and $\vee$-E.

Crucial deficiencies of **C**, **I** and **M**

Here are two places where Gentzen and Prawitz permit vacuous discharge of assumptions:

$$(\neg\text{-I}) \quad \dfrac{\begin{array}{c} \diamond\overline{\ \ \varphi\ \ }^{(i)} \\ \vdots \\ \bot \end{array}}{\neg\varphi}(i) \qquad\qquad (\text{CR}) \quad \dfrac{\begin{array}{c} \diamond\overline{\ \ \neg\varphi\ \ }^{(i)} \\ \vdots \\ \bot \end{array}}{\varphi}(i)$$

These yield, respectively, the negative and positive forms of the First Lewis Paradox.

Classical Reductio with vacuous discharge is none other than EFQ!

Crucial deficiencies of **C**, **I** and **M**

Gentzen and Prawitz do not register *in the form of a primitive inference* the fact that the truth table for $\rightarrow$ tells us that any conditional with a false antecedent is true. This can (and should) receive *direct inferential expression* as a part of the introduction rule for $\rightarrow$:

$$(\rightarrow\text{-I}) \qquad \dfrac{\begin{array}{c} \dfrac{\square}{\varphi}(i) \\ \vdots \\ \underline{\bot} \end{array}}{\varphi \rightarrow \psi}(i)$$

This can be maintained by the relevantist, who eschews Ex Falso Quodlibet. (NB: The intuitionist cannot *object* to this rule, since it is *derivable* in Intuitionistic Logic.)

Crucial deficiencies of **C**, **I** and **M**

The proof of Disjunctive Syllogism should *not* have to resort to Ex Falso Quodlibet. That is why $\vee$-E, or Proof by Cases, should *not* require equiform subordinate conclusions, the way it does in the Gentzen–Prawitz natural-deduction systems. One should be able to effect a proof by cases in either of the following two ways:

$$\dfrac{\varphi \vee \psi \qquad \begin{array}{c} {}^{\square}\overline{\varphi}^{(i)} \\ \vdots \\ \bot \end{array} \qquad \begin{array}{c} {}^{\square}\overline{\psi}^{(i)} \\ \vdots \\ \theta \end{array}}{\theta}(i) \qquad\qquad \dfrac{\varphi \vee \psi \qquad \begin{array}{c} {}^{\square}\overline{\varphi}^{(i)} \\ \vdots \\ \theta \end{array} \qquad \begin{array}{c} {}^{\square}\overline{\psi}^{(i)} \\ \vdots \\ \bot \end{array}}{\theta}(i)$$

Mantra for proof by cases:

> *If either one of your case-proofs ends with absurdity, you may bring down as your main conclusion the conclusion of the other case.*

Allowing →-Introduction to *discharge* an assumption

Note that we have thus far obtained an introduction rule for → in only the two parts that correspond to the respective considerations that it suffices, for the truth of a conditional, to have its antecedent false, or to have its consequent true. But this leaves out of the picture the most familiar form of →-Introduction, known as Conditional Proof, which allows for the discharge of the antecedent φ as an assumption if it has been used in the derivation of the consequent ψ as the conclusion of the subordinate proof:

$$\dfrac{\begin{array}{c}\overline{\varphi}^{(i)}\\ \vdots \\ \psi\end{array}}{\varphi \rightarrow \psi}\,(i)$$

Rules of Deduction: An extra part for $\rightarrow$-Introduction

The first two parts of $\rightarrow$-Introduction immediately below were obtained by morphing the rule of $\rightarrow$-Verification. The third part, allowing discharge of assumptions, can now be added to them, in light of the derivations below.

$$
\dfrac{\begin{array}{c} \underbrace{\overset{\square\,\overline{\qquad}^{(i)}}{\varphi}\ ,\ \Delta} \\ \Pi \\ \bot \end{array}}{\varphi \rightarrow \psi}\,(i)
\qquad
\dfrac{\begin{array}{c} \Delta \\ \Pi \\ \psi \end{array}}{\varphi \rightarrow \psi}
\qquad
\dfrac{\begin{array}{c} \underbrace{\overset{\overline{\qquad}^{(i)}}{\varphi}\ ,\ \Delta} \\ \Pi \\ \psi \end{array}}{\varphi \rightarrow \psi}\,(i)
$$

The following derivations employ **Dilemma**—the first on the antecedent A as positive horn-assumption, the second on the consequent B as the same:

$$
\dfrac{\dfrac{\begin{array}{c} {}^{(2)}\overline{A} \\ \vdots \\ B \end{array}}{A \rightarrow B}
\qquad
\dfrac{\dfrac{{}^{(2)}\overline{\neg A} \qquad \overline{A}^{(1)}}{\bot}}{A \rightarrow B}\,(1)}{A \rightarrow B}\,(2)
\qquad
\dfrac{\dfrac{{}^{(2)}\overline{B}}{A \rightarrow B}
\qquad
\dfrac{\dfrac{{}^{(2)}\overline{\neg B} \qquad \begin{array}{c} \overline{A}^{(1)} \\ \vdots \\ B \end{array}}{\bot}}{A \rightarrow B}\,(1)}{A \rightarrow B}\,(2)
$$

Portmanteau graphic statement of $\to$-Introduction

We now have the following three 'parts' of the rule of $\to$-Introduction:

$$\begin{array}{c} \Box\overline{}^{(i)} \\ \underbrace{\varphi\ ,\ \Delta} \\ \Pi \\ \dfrac{\bot}{\varphi\to\psi}\,(i) \end{array} \qquad \begin{array}{c} \Delta \\ \Pi \\ \dfrac{\psi}{\varphi\to\psi} \end{array} \qquad \begin{array}{c} \overline{}^{(i)} \\ \underbrace{\varphi\ ,\ \Delta} \\ \Pi \\ \dfrac{\psi}{\varphi\to\psi}\,(i) \end{array}$$

The second and third of these can be melded so that the display becomes

$$\begin{array}{c} \Box\overline{}^{(i)} \\ \underbrace{\varphi\ ,\ \Delta} \\ \Pi \\ \dfrac{\bot}{\varphi\to\psi}\,(i) \end{array} \qquad \begin{array}{c} \Diamond\overline{}^{(i)} \\ \underbrace{\varphi\ ,\ \Delta} \\ \Pi \\ \dfrac{\psi}{\varphi\to\psi}\,(i) \end{array}$$

The diamond in the graphic rule-part on the *right* says 'vacuous' discharge is permitted: φ need not be an undischarged assumption of the subordinate proof Π. This is exactly the rule of ($\to$-I) in the Gentzen–Prawitz tradition. The graphic rule-part on the *left* is a 'new part' of ($\to$-I) supplied by the Core logician, via the foregoing method of morphing. It directly respects the third and fourth lines of the truth table for $\to$. It is derivable in the Gentzen–Prawitz system by appeal to the rule EFQ, *which the Core logician eschews*.

Let me now state the rules of natural deduction that result from these investigations that are motivated by the desire to avoid irrelevance but still be able to regiment all deductive reasoning in mathematics and science.

(¬-I)

$$\dfrac{\begin{array}{c}\overline{\square}^{\,(i)} \\ \varphi \\ \vdots \\ \bot\end{array}}{\neg\varphi}\,(i)$$

(¬-E)

$$\dfrac{\neg\varphi \qquad \begin{array}{c}\vdots \\ \varphi\end{array}}{\bot}$$

Graphic Rules for Core Logic

(∧-I)

$$\dfrac{\begin{matrix}\vdots & & \vdots \\ \varphi & & \psi\end{matrix}}{\varphi \land \psi}$$

(∧-E)

$$\dfrac{\varphi \land \psi \qquad \begin{matrix} {}^{(i)}\text{——}\square\text{——}{}^{(i)} \\ \underbrace{\varphi\ ,\ \psi} \\ \vdots \\ \theta \end{matrix}}{\theta}\,(i)$$

Graphic Rules for Core Logic

(∨-I)

$$\cfrac{\begin{array}{c}\vdots\\ \varphi\end{array}}{\varphi \color{red}{\vee} \psi} \qquad \cfrac{\begin{array}{c}\vdots\\ \psi\end{array}}{\varphi \color{red}{\vee} \psi}$$

(∨-E)

$$\cfrac{\varphi \color{blue}{\vee} \psi \quad \begin{array}{c}\overline{\square}^{(i)}\\ \varphi\\ \vdots\\ \theta\end{array} \quad \begin{array}{c}\overline{\square}^{(i)}\\ \psi\\ \vdots\\ \theta\end{array}}{\theta}(i) \qquad \cfrac{\varphi \color{blue}{\vee} \psi \quad \begin{array}{c}\overline{\square}^{(i)}\\ \varphi\\ \vdots\\ \bot\end{array} \quad \begin{array}{c}\overline{\square}^{(i)}\\ \psi\\ \vdots\\ \theta\end{array}}{\theta}(i) \qquad \cfrac{\varphi \color{blue}{\vee} \psi \quad \begin{array}{c}\overline{\square}^{(i)}\\ \varphi\\ \vdots\\ \theta\end{array} \quad \begin{array}{c}\overline{\square}^{(i)}\\ \psi\\ \vdots\\ \bot\end{array}}{\theta}(i)$$

Graphic Rules for Core Logic

(→-I)

$$\dfrac{\begin{array}{c}\square\!\!-\!\!-(i)\\ \varphi\\ \vdots\\ \bot\end{array}}{\varphi \rightarrow \psi}\,(i) \qquad \dfrac{\begin{array}{c}\diamond\!\!-\!\!-(i)\\ \varphi\\ \vdots\\ \psi\end{array}}{\varphi \rightarrow \psi}\,(i)$$

(→-E)

$$\dfrac{\varphi \rightarrow \psi \qquad \begin{array}{c}\vdots\\ \varphi\end{array} \qquad \begin{array}{c}\square\!\!-\!\!-(i)\\ \psi\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

Graphic Rules for Core Logic

(∃-I)

$$\frac{\begin{array}{c}\vdots\\ \varphi^x_t\end{array}}{\exists x\varphi}$$

(∃-E)

$$\frac{\exists x\varphi \; \not{a} \qquad \begin{array}{c}\overset{\square\!\!-\!\!-(i)}{\underbrace{\not{a}\;\ldots\;\varphi^x_a\;\ldots\;\not{a}}}\\ \vdots\\ \psi \; \not{a}\end{array}}{\psi}\,(i)$$

Graphic Rules for Core Logic

(∀-I)

$$\dfrac{\begin{array}{c} \not{a} \\ \vdots \\ \varphi \end{array}}{{\color{red}\forall} x \varphi^a_x}$$

(∀-E)

$$\dfrac{{\color{blue}\forall} x\varphi \qquad \begin{array}{c} \underbrace{\overset{(i)}{\overline{\varphi^x_{t_1}}} \;,\; \ldots \square \ldots \;,\; \overset{(i)}{\overline{\varphi^x_{t_n}}}} \\ \vdots \\ \theta \end{array}}{\theta}\,(i)$$

One obtains the system $\mathbb{C}^+$ of Classical Core Logic by adding to $\mathbb{C}$ suitably relevantized versions of *Classical Reductio* or *Dilemma*:

$$
(\mathrm{CR})\qquad \dfrac{\begin{array}{c} \Box\,\overline{}\,(i) \\ \neg\varphi \\ \vdots \\ \bot \end{array}}{\varphi}\,(i)
\qquad\qquad
(\mathrm{Dil})\qquad \dfrac{\begin{array}{c} \Box\,\overline{}\,(i) \\ \varphi \\ \vdots \\ \psi \end{array} \qquad \begin{array}{c} \Box\,\overline{}\,(i) \\ \neg\varphi \\ \vdots \\ \psi/\bot \end{array}}{\psi}\,(i)
$$

Core deducibility

Definition

Unless otherwise indicated we shall assume, of the deducibility relation $\vdash$, that it satisfies no more than the rules of the system $\mathbb{C}$ of Core Logic. (To be precise, $\Delta \vdash \varphi$ means that φ is core-deducible from premises that are drawn from Δ. The set of premises in question may be a proper subset of Δ.)

Definition

We write Δ_1, Δ_2 for $\Delta_1 \cup \Delta_2$, and we write Δ, φ for $\Delta \cup \{\varphi\}$.

Cut Admissibility for Core Proof

Theorem (Cut Admissibility for Core Proof)

There is an effective method $[\ ,\ ]$ *that transforms any two core proofs*

$$\begin{array}{ccc} \Delta & \varphi, \Gamma & \\ \Pi & \Sigma & \textit{(where } \varphi \notin \Gamma \textit{ and } \Gamma \textit{ may be empty)} \\ \varphi & \theta & \end{array}$$

into a core proof $[\Pi, \Sigma]$ of θ or of $\bot$ *from (some subset of)* $\Delta \cup \Gamma$.

Proof. See Neil Tennant, 'Cut for Core Logic', *Review of Symbolic Logic*, 5, no. 3, 2012, pp. 450–479. DOI: 10.1017/S1755020311000360; and 'Cut for Classical Core Logic', *Review of Symbolic Logic*, 8, no. 2, 2015, pp. 236–256. DOI: http://dx.doi.org/10.1017/S1755020315000088.

Note the potential 'subsetting down' *on the left* **and** *on the right*.

Cut with potential Epistemic Gain

Corollaries:

CUT WITH POTENTIAL EPISTEMIC GAIN
If $\Delta \vdash \varphi$ and $\Gamma, \varphi \vdash \psi$, then either $\Delta, \Gamma \vdash \bot$ or $\Delta, \Gamma \vdash \psi$.

CUT FOR ABSURDITY (or 'CUT for $\bot$')
If $\Delta \vdash \varphi$ and $\Gamma, \varphi \vdash \bot$, then $\Delta, \Gamma \vdash \bot$.

CUT ON CONSISTENT PREMISES
If $\Delta \nvdash \bot$, and $\Delta \vdash \theta$ for each $\theta \in \Gamma$, and $\Gamma \vdash \psi$, then $\Delta \vdash \psi$.

Results about relationships between systems

Theorem

If $\Delta \vdash_I \varphi$, *then for some* $\Gamma \subseteq \Delta$, *either* $\Gamma \vdash \varphi$ *or* $\Gamma \vdash \bot$.

Theorem

If $\Delta \vdash_C \varphi$, *then for some* $\Gamma \subseteq \Delta$, *either* $\Gamma \vdash_{\mathbb{C}^+} \varphi$ *or* $\Gamma \vdash_{\mathbb{C}^+} \bot$.

Theorem

If $\Delta \vdash_C \varphi$, *then* $\forall\neg\neg[\neg\neg\Delta] \vdash \forall\neg\neg[\neg\neg\varphi]$ *or* $\forall\neg\neg[\neg\neg\Delta] \vdash \bot$.

Philosophical and Methodological Observations

- The so-called 'loss' of *unrestricted* transitivity of deduction in Core Logic $\mathbb{C}$ and in Classical Core Logic $\mathbb{C}^+$ brings with it *epistemic gain*. One has transitivity *everywhere one needs it*.
- Core Logic $\mathbb{C}$ suffices for Constructive Mathematics
- Classical Core Logic $\mathbb{C}^+$ suffices for Classical Mathematics
- Core Logic $\mathbb{C}$ suffices for hypothetico-deductive testing of scientific theories
- The *same method of constructivizing* gives $\mathbf{I} \subset \mathbf{C}$ and $\mathbb{C} \subset \mathbb{C}^+$.
- The *same method of relevantizing* gives $\mathbb{C} \subset \mathbf{I}$ and $\mathbb{C}^+ \subset \mathbf{C}$
- Core Logic $\mathbb{C}$ suffices for the reasoning involved in the logical and semantic paradoxes

Philosophical and Methodological Observations (cont.)

- *Natural deductions* and *sequent proofs* are structurally isomorphic (in $\mathbb{C}$ and in $\mathbb{C}^+$). This is on account of our new method of formulating ND (parallelized elimination rules, with MPEs standing proud), rather than the particularities of the Core systems. *This innovation was motivated by the pursuit of computationally efficient proof-search in* $\mathbb{C}$. *See Autologic, Edinburgh University Press, 1992.*
- As we have seen, $\mathbb{C}$ and $\mathbb{C}^+$ are obtained by smoothly generalizing model-*relative* rules for verification and falsification to model-*invariant* natural-deduction rules of introduction and elimination, respectively. It is this method of generalization ('morphing') that ensures the unusual and striking features of Core natural deductions: parallelized elimination rules with MPEs standing proud; all proofs being in normal form; obligatory discharging of certain assumptions; liberalized $\rightarrow$-I and $\vee$-E; and eschewal of EFQ.

Philosophical and Methodological Observations (cont.)

- Core Logic $\mathbb{C}$ is the logic of 'conceptual constitution': it suffices for neo-logicist derivation of number-theoretic axioms from deeper logical principles governing the operator $\#x\Phi(x)$ (the number of Φs)
- Core Logic $\mathbb{C}$ is the *minimal inviolable core* of logic that is needed, and suffices, for rational belief revision
- Classical Core Logic $\mathbb{C}^+$, hence also Core Logic $\mathbb{C}$, enjoys the strongest possible 'variable-sharing' result. *This affords a strong 'relevance filter' for computational proof-search.*
- There is an automated deducer for (propositional) $\mathbb{C}$, whose decision-problem is PSPACE-complete (like that of $\mathbf{I}$).
- Automated proof-search in $\mathbb{C}$ is sped up by (i) relevance filtration, and (ii) non-forfeiture of epistemic gain.

Some recent developments concerning $\mathbb{C}$ and $\mathbb{C}^+$

1. Ethan Brauer has proved (*RSL*, 2020) that proofs in $\mathbb{C}$ and $\mathbb{C}^+$ can be coarsened so that every subproof is a proof of a perfectly valid or a gauntly valid sequent.
2. The present author has argued (*Philosophical Quarterly*, 2022) that $\mathbb{C}^+$ is a clear counterexample to Timothy Williamson's claim (*Philosophical Issues*, 2018) that substructural logics are 'ill-suited to acting as background logics for science'.
3. The present author has argued (to the Ohio State Logic Group, February 2022) that the Core logician can profitably explore the possibility of modifying the standard relation $\models$ of logical consequence to a relation $\Vert\!\!\models$ of *genuine* logical consequence, according to which any unsatisfiable set of sentences has *only* $\bot$ as a genuine logical consequence.
   In a nutshell: replace *Explosion* with *Implosion*!

Let us explore now how to re-fashion the concept of logical consequence so that it is *implosive* rather than *explosive*.

Core Logical Consequence $\Vert\!\models$

Here is an *inferentialist* metalinguistic definition of the concept $\Vert\!\models$ (of *core logical consequence*). The sortal parameter M for models in the introduction rules and the elimination rule E_2 is may occur only where indicated. We presume a prior grasp of the relation $\Vdash$ of a model making a sentence true.

$\Vert\!\models$-**I**
$$\dfrac{\begin{array}{c}\overline{M \Vdash \Delta}^{(i)}\\ \vdots \\ \underline{\perp}\end{array}}{\Delta \Vert\!\models \perp}{(i)}$$

$\Vert\!\models$-E_1
$$\dfrac{\Delta \Vert\!\models \perp \qquad \mathbf{M} \Vdash \Delta}{\underline{\perp}}$$

$\Vert\!\models$-**I**
$$\dfrac{\mathbf{M} \Vdash \Delta \qquad \begin{array}{c}\overline{M \Vdash \Delta}^{(i)}\\ \vdots \\ M \Vdash \varphi\end{array}}{\Delta \Vert\!\models \varphi}{(i)}$$

$\Vert\!\models$-E_2
$$\dfrac{\Delta \Vert\!\models \varphi \qquad \begin{array}{c}\overline{M \Vdash \Delta}^{(i)}\\ \vdots \\ \psi\end{array}}{\psi}{(i)}$$

$\Vert\!\models$-E_3
$$\dfrac{\Delta \Vert\!\models \varphi \qquad \mathbf{M} \Vdash \Delta}{\mathbf{M} \Vdash \varphi}$$

EFQ is super-fallacious

Lemma

EFQ is super-fallacious.

Proof.

For the application of $\VDash$-E_2 in the formal metalinguistic proof below we are taking Δ to be the singleton of falsum:

$$\dfrac{\dfrac{{\scriptstyle(2)}\dfrac{}{\bot \VDash \varphi} \qquad \dfrac{\dfrac{}{M \Vdash \bot}{\scriptstyle(1)}}{\bot}{\scriptstyle(\alpha)}}{\bot}{\scriptstyle(1)\ \VDash\text{-E}_2}}{\bot \nVDash \varphi}{\scriptstyle(2)}$$

□

Lewis's First Paradox is fallacious.

Lemma

It is not the case that $A, \neg A \parallel\!\models B$.

Proof.

$$\dfrac{A, \neg A \parallel\!\models B \qquad \dfrac{\dfrac{\dfrac{\overline{M \Vdash \{A, \neg A\}}^{(1)}}{M \Vdash \neg A}}{M \nVdash A} \qquad \dfrac{\overline{M \Vdash \{A, \neg A\}}^{(1)}}{M \Vdash A}}{\underline{\bot}}}{\underline{\bot}}\;(1)\ \parallel\!\models\text{-E}_2$$

□

The relation $\Vert\!\models$ is transitive in the following sense.

Lemma

$$\left.\begin{array}{r}\Delta \Vert\!\models \varphi \\ \varphi, \Gamma \Vert\!\models \psi \\ \Delta, \Gamma \ \textit{satisfiable}\end{array}\right\} \Rightarrow \ \Delta, \Gamma \Vert\!\models \psi.$$

Proof.

$$\dfrac{\exists M\, M \Vdash \Delta, \Gamma \qquad \dfrac{\dfrac{}{\mu \Vdash \Delta, \Gamma}^{(2)} \qquad \dfrac{\varphi, \Gamma \Vert\!\models \psi \qquad \dfrac{\dfrac{\Delta \Vert\!\models \varphi \qquad \dfrac{\dfrac{}{M \Vdash \Delta, \Gamma}^{(1)}}{M \Vdash \Delta}}{M \Vdash \varphi}\ \Vert\!\models\!\mathsf{E}_3 \qquad \dfrac{\dfrac{}{M \Vdash \Delta, \Gamma}^{(1)}}{M \Vdash \Gamma}}{M \Vdash \varphi, \Gamma}}{M \Vdash \psi}\ \Vert\!\models\!\mathsf{E}_3}{\Delta, \Gamma \Vert\!\models \psi}\ (1)\ \Vert\!\models\!\text{-I}}{\Delta, \Gamma \Vert\!\models \psi}\,(2)$$

$\square$

Let us see now just how intimately relevant are premises to conclusions of Classical Core proofs.

Relevance

Definition

$\varphi \underset{A}{\approx} \psi \equiv_{\text{df}}$ the atom A has occurrences of the *same* parity in φ and in ψ

Definition

$\varphi \approx \psi \equiv_{\text{df}}$ for some atom A we have $\varphi \underset{A}{\approx} \psi$.

Definition

$\varphi \approx \Delta \equiv_{\text{df}}$ for some ψ in Δ, we have $\varphi \approx \psi$

Relevance

Definition

$\varphi \underset{A}{\bowtie} \psi \equiv_{\mathrm{df}}$ the atom A has occurrences of *opposite* parities in φ and in ψ

Definition

$\varphi \bowtie \psi \equiv_{\mathrm{df}}$ for some atom A we have $\varphi \underset{A}{\bowtie} \psi$.

Definition

$\pm\varphi \equiv_{\mathrm{df}} \varphi \bowtie \varphi$.

Relevance

Definition

A sequence $\varphi_1, \ldots, \varphi_n$ $(n > 1)$ of pairwise distinct sentences is a $\bowtie$*-path connecting* φ_1 *to* φ_n *in* Δ $\equiv_{\text{df}}$ for $1 \leq i \leq n$, φ_i is in Δ, and for $1 \leq i < n$, $\varphi_i \bowtie \varphi_{i+1}$.

Definition

φ and ψ are $\bowtie$*-connected in* Δ (in symbols: $\varphi \underline{\bowtie} \psi$) $\equiv_{\text{df}}$ if $\varphi \neq \psi$, then there is a $\bowtie$-path connecting φ to ψ in Δ.

Definition

A set Δ of formulae is $\bowtie$*-connected* $\equiv_{\text{df}}$ for all φ, ψ in Δ, if $\varphi \neq \psi$, then $\varphi \underline{\bowtie} \psi$.

Definition

A $\bowtie$-*component* of Δ is a non-empty, inclusion-maximal $\bowtie$-connected subset of Δ (where the $\bowtie$-connections are established via members of Δ).

Definition

$\varphi \blacktriangleleft \Delta \quad \equiv_{\text{df}} \quad$ for every $\bowtie$-component Γ of Δ, $\varphi \approx \Gamma$.

Definition

Suppose $\Delta \neq \emptyset$. Then

$$\natural\Delta \;\equiv_{\text{df}}\; \begin{cases} \text{if } \Delta \text{ is a singleton, say } \{\delta\}, \text{ then } \pm\,\delta; \\ \text{ and} \\ \text{if } \Delta \text{ is not a singleton, then } \Delta \text{ is } \bowtie\text{-connected} \end{cases}$$

Definition

We shall say that a set Δ of premises is *relevantly connected both within itself and to a conclusion* φ (in symbols: $\mathcal{R}(\Delta, \varphi)$) just in case exactly one of the following three conditions is satisfied:

1. Δ is non-empty, φ is $\bot$, and $\natural\Delta$.
2. Δ is non-empty, φ is not $\bot$, and $\varphi \blacktriangleleft \Delta$.
3. Δ is empty, φ is not $\bot$, and $\pm\,\varphi$.

Theorem: For every Classical Core proof of φ whose premises form the set Δ, we have $\mathcal{R}(\Delta, \varphi)$.

$\mathcal{R}(\Delta, \varphi)$	$\varphi = \bot$	$\varphi \neq \bot$
$\Delta = \emptyset$		$\pm\,\varphi$ Case 3
$\Delta \neq \emptyset$	$\natural\Delta$ Case 1	$\varphi \blacktriangleleft \Delta$ Case 2

The crucial epistemological consideration

Proofs afford *certainty*. They are effectively checkable. Proofs are *sound*. They guarantee that truth is transmitted from their premises to their conclusions. Thus: if there is a proof of φ from Δ, then any model of Δ makes φ true.

It follows that if we are presented with a proof of φ from Δ, then we *can know with certainty* that any model of Δ makes φ true; and if we are presented with a proof of $\perp$ from Δ, then we *can know with certainty* that Δ has no model.

When Δ is a set of axioms for a mathematical theory, we can know with certainly, of any sentence φ for which we have a proof from Δ, that it is true in any model of Δ. In particular, if we are theorizing about a unique model M by framing axioms Δ true of M, and we prove φ from Δ, then we know with certainty that φ is true in M.

The crucial epistemological consideration

All that is needed for our system $\mathcal{S}$ of proof to be an instrument for expansion of mathematical knowledge is *soundness* and *effective checkability* of proof, along with *transitivity of logical consequence in the language* for which $\mathcal{S}$ is the system of proof. We do *not* need the deductive rule of CUT as a rule *in* our system $\mathcal{S}$ of proof. It suffices that cut be *admissible* for $\mathcal{S}$, in the following sense:

$$(\Delta \vdash_{\mathcal{S}} \varphi \text{ and } \varphi, \Gamma \vdash_{\mathcal{S}} \psi) \;\; \Rightarrow \;\; (\Delta, \Gamma \vdash_{\mathcal{S}} \psi \text{ or } \Delta, \Gamma \vdash_{\mathcal{S}} \bot).$$

In Core Logic, as already explained, we have something even better. There is an effective binary function $[\Pi, \Sigma]$ on $\mathbb{C}^+$-proofs such that

$$\left[\begin{array}{ccc} \Delta & & \varphi, \Gamma \\ \Pi & , & \Sigma \\ \varphi & & \psi \end{array}\right] \text{ is a } \mathbb{C}^+\text{-proof of } \psi \text{ or of } \bot \text{ from some subset of } \Delta \cup \Gamma.$$

The inferentialist (at the meta-level) can capture the notion $\models$ of standard logical consequence. The inferentialist supplies for it an introduction rule and an harmoniously balancing elimination rule. In the rule $\models$-I, 'M' is a sortal parameter for models, occurring only where indicated ; in the rule $\models$-E, '**M**' is a sortal term for a model.

$$\models\text{-I} \qquad \dfrac{\dfrac{}{M \Vdash \Omega}^{(i)} \quad \begin{matrix} \vdots \\ M \Vdash \theta \end{matrix}}{\Omega \models \theta}{}_{(i)} \qquad\qquad \models\text{-E} \qquad \dfrac{\Omega \Vdash \theta \quad \mathbf{M} \Vdash \Omega}{\mathbf{M} \Vdash \theta}$$

With the foregoing rules, Core metalogic proves transitivity of $\models$:

$$\frac{\Delta \models \varphi \qquad \varphi, \Gamma \models \psi}{\Delta, \Gamma \models \psi}\ .$$

Proof:

$$\dfrac{\dfrac{\varphi, \Gamma \models \psi \qquad \dfrac{\dfrac{\Delta \models \varphi \qquad \dfrac{\overline{M \Vdash \Delta, \Gamma}^{(1)}}{M \Vdash \Delta}}{M \Vdash \varphi}{\scriptstyle \models\text{-E}} \qquad \dfrac{\overline{M \Vdash \Delta, \Gamma}^{(1)}}{M \Vdash \Gamma}}{M \Vdash \varphi, \Gamma}}{M \Vdash \psi}{\scriptstyle \models\text{-E}}}{\Delta, \Gamma \models \psi}{\scriptstyle (1)\models\text{-I}}$$

Neil Tennant.
A Logical Theory of Truthmakers and Falsitymakers.
In Michael Glanzberg, editor, *Handbook on Truth*, pages 355–393. Oxford University Press, Oxford, 2018.