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FROM WORLDS TO POSSIBILITIES

1. INTRODUCTION

The purpose of the discussion to follow is that of showing how to do for modal logic a job somewhat similar to that done for tense logic in [15]. In that paper I attempted to indicate, picking up some suggestions of C. L. Hamblin, what becomes of standard tense logic in the style of Prior, when instants or moments of time are replaced by intervals or periods of time as the temporal entities with respect to which formulae are evaluated for truth.[1] Here an interval is taken as an entity *sui generis*, rather than as a set of moments of time (satisfying various further conditions). While this is not essential from the point of view of the formal developments, if such a perspective is taken, those developments may be seen as one aspect of the elaboration of an Aristotelian view in the philosophy of time, according to which there are actually no such things as dimensionless temporal points – at least, as standardly construed (i.e., as exhaustively constituting temporal intervals). Rather, on this view, only such talk of instants and of what is true at them as is definable in terms of intervals and of what is true over them is to be regarded as legitimate. There are two (doubtless combinable) views one may take of instants once a primary status has been accorded to intervals in this way. On the first, instants are essentially boundaries between intervals (and so, for example, may be defined by construction out of adjacent intervals[2]). On the second, instants are limits of nested sequences of subintervals, and talk which takes them too seriously is diagnosed as involving what has been called the 'infinitieth term' fallacy. The only temporal entities robust enough for talk of truth with respect to them to be primitively intelligible (which is how I incline to interpret traditional formulations like 'the only genuine *parts* time has') are *intervals*, and the salient fact about interval subdivision is that it is a process which does not terminate. As we shall see, it is this second strand in the philosophical opposition to instants which has an analogue in the theory of modality, seeking to replace possible worlds with something looking less suspiciously like the terminations of unending processes.

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Now, why might anyone have qualms of the envisaged sort about possible worlds? It is clear that they are not to be thought of as part and parcel of the usual type of philosophical opposition to what has come to be called realism about possible worlds, which has its source in the idea that the realist makes an important mistake in passing from the premises that things *might be* a certain way, to the conclusion that with respect to a something-or-other, things *are* that way.[3] This mistake will be felt to have been comitted whether the something-or-other is a *point* in logical space (a possible world), to employ a pictorially suggestive way of speaking, or a *region* of logical space not regarded as a set of such points.[4] (Similarly, an opposition of the kind alluded to above to moments of time on the grounds of their objectionable punctiformity which would urge their replacement by temporal intervals has nothing to do with that more general opposition to either sort of temporal entity as an unwarranted substantivalizing of tensed discourse.) Nevertheless, our introductory purposes are well served by looking at one reaction that has been aroused by a part of David Lewis's famous defence of realism in [18]. Lewis begins with a preliminary observation that ordinary speech reveals a *prima facie* ontological commitment to the entities being defended:

> I believe that there are possible worlds other than the one we happen to inhabit. If an argument is wanted, it is this. I believe, and so do you, that things could have been different in counltess ways. But what does this mean? Ordinary language permits the paraphrase: there are many ways things could have been besides the way they actually are. On the face of it, this sentence is an existential quantification. It says that there exist many entities of a certain description, to wit 'ways things could have been'. I believe that things could have been different in countless ways; I believe permissable paraphrases of what I believe; taking the paraphrase at its face value, I therefore believe in the existence of entities that might be called 'ways things could have been'. I prefer to call them 'possible worlds'.[5]

This discussion evoked an interesting response from M. K. Davies, who in [5] writes:

> Aside form the point that the admissibility of the paraphrase in the opposite direction constitutes a *prima facie* case against any *need* to recognize possible worlds, it is clear, since a way things could have been, must be, for Lewis, a way *everything* could have been, and so cannot be specifiable in a single sentence, that this is *no* paraphrase. At most it could show how we recognize ways things might have been dfferent, and examples of these can be given in single sentences. I might have had straight hair. That is a way things might have been different, but it does not specify a *way things might have been*, in the sense Lewis needs.[6]

Here we have a motivation for the pursuit of modal logic against a semantic background in which less determinate entities than possible worlds, things which I am inclined for want of a better word to call simply *possibilities*, are what sentences (or fomulae) are true or false with respect to. Our entrée into the realm of the possible, one might say, is though imagining and story-telling about how things might be, considering what might happen under hypothetical circumstances, entertaining a counterfactual supposition, and so on; but each such introduction into that realm presents us what I called above a region of logical space, and not with a point thereof. These points, however, are usually quite easy to sell, once the 'regions' idea has been got across, because one accepts as an idealization the concept of a fully specified possibility, determinate in every respect. But one might, on the other hand, not like this idealization, since one might hold the position that a possibility can always be further specified, just as a temporal interval can always be further subdivided. On such a view of the matter – which I have, of course, merely outlined and not attempted to argue for – the idea of a possible world becomes, in general, inadmissible in purporting to be the descriptum of a fully specific description of how things might be while there can actually be no 'fully specific' description, every characterization of a possibility being capable of being further specified. (I say that the idea of a possible world is *in general* inadmissible in anticipation of the response: but surely you acknowledge that the actual world, at least, is a fully determinate possibility: *its* punctiformity is not open to question. Fortunately we do not need to decide on the question of whether or not this response is correct before looking at the semantic repercussions of replacing worlds by less specific possibilities, since the notion of a model with which we shall be operating does not especially distinguish any one such possibility as 'actual'. Presumably the view that the actual world is fully determinate in this or that respect is the view that a realistic interpretation, in Dummett's sense,[7] is correct for the relevant class of statements.)

We may now distinguish a weaker and a stronger suggestion distillable from these considerations. The weaker suggestion is that we should *allow* the entities with respect to which truth is evaluated to be not fully determinate, the stronger suggestion being that we *require* them to be thus indeterminate. In what follows, it is the weaker suggestion I take up, which leaves open the option of imposing a special condition on models which gives expression to the stronger suggestion, a matter which will receive occasional comment below (beginning with note 9).

I want to conduct the investigation under the following constraint: the object-language under investigation should be just the familiar language of standard modal logic. It is a question of showing how to interpret this language without the aid of possible worlds, though still in an extensional metalanguage in the customary model-theoretic manner: not of tampering with the language in any way to make it more tractable, or to record new distinctions especially salient from the novel semantical point of view. This marks a departure from the approach of [15], in which a standard tense logical language is equipped with *two* negation operators – one weak or classical-style negation for forming a denial that the formula negated is true over a temporal interval, and another strong or intuitionist-style negation for forming a denial the formula negated is true over any subinterval of the interval in question. When we allow ourselves only a single negation connective, it seems appropriate to interpret it in this second, stronger sort of way, since this decision allows us to make the usual identification of the falsity of a statement with the truth of its negation, while simultaneously acknowledging that to say that something would be false were a certain possibility to be realized (which is how the '$X \models \sim \alpha$' of the following section may be read) is to say that however that possibility is realized – i.e., under any further specification, or, as I shall say, *refinement* thereof, it would be false. There is a cost, though; with respect to this conception of falsity, bivalence fails: formulae will often fail to be either true or false relative to certain possibilities. (Nevertheless, it would, for the usual sorts of reasons, be quite misleading to say that we have here a three-valued logic on our hands.) Actually, by slight changes in terminology and notaion, bivalence can be preserved, if we use 'false' to mean 'not true' – so that $V(\alpha)$, in Section 2, is never undefined (and we do not require a false formula to remain false on further refinements). That would bring the present treatment more into line with the presentation in [15] – though I have opted for the current way of putting matters because it seems most natural in the present modal setting.[8]

On the subject of comparisons, I should mention the work of J. A. W. Kamp [16] and, especially, K. Fine [12], on supervaluational treatments of vagueness. (Fine's paper may be consulted for the purpose of putting the material in Section 2 in a more general setting.) The similarity between such treatments of vagueness and the present account of indeterminate possibilities is considerable. Taking truth *simpliciter* for vague sentences as truth on all admissible precisifications of the vagueness involved is like having truth on a possibility coincide – as it does in the present theory – with truth on every refinement of that possibility. However, the leading idea behind any supervaluations treatment is that of relating truth on a bivalent valuation (the primary concept inductively defined for formulae) to truth *simpliciter* by means of universal quantification, whereas on the present account there is no question of such a modulation from one to another notion of truth since we have only the single concept of truth in (or 'at' or 'over') a possibility. At an intuitive level, one feels that the vagueness of:

> Harriet crossed the equator in the late afternoon of January 1, 1980.

occasioned (assuming the reference of 'Harriet' is clear) by the vague phrase 'late afternoon' is quite different from the 'modal' indeterminacy of:

> Harriet crossed the equator on January 1, 1980.

resulting from that sentence's leaving open such things as the exact time of the crossing, the colour of Harriet's scarf at that (or any other) time, and the details of what was going on then in Tokyo, for example. Every sentence of a natural language will exhibit this sort of indeterminacy, regardless of how free of imprecise expressions it may be: it quite precisely singles out a region, rather than a point, of logical space, so that there is no question – as there always is with vagueness – of the existence of borderline cases. (I suppose one *could* introduce into English sentences functioning like the world-propositions of Meredith and Prior [21], which would necessitate a revision of the generalization about natural languages just made.) One might, all the same, if working with the notions both of truth at a world and truth over a region of logical space, or 'possibility', use a supervaluational approach to relate them. But the effort of the present investigations is, of course, to do without the former notion altogether. Finally, I should like to remark that what follows is intended more as an opening up of this sort of territory than as something I regard as definitively the one and only correct way of proceeding with its exploration.

2. NON-MODAL CONNECTIVES

We want to show how to do model theory for the familiar systems of modal logic without appealing to collections of possible worlds. A model will be a quadruple $\langle W, \geqslant, R, V\rangle$ in which W is a non-empty set of entities we call *possibilities* (and over which we use 'X', 'Y', . . . as variables, the use of upper case latters here to contrast with the customary use of lower case letters for possible worlds) on which $\geqslant$ and R are binary relations, and V is a partial function taking propositional variables paired with elements of W to truth-values. The relation R is to play the usual role of the accessibility relation in modal logic. In general, we place no special conditions on this relation; we shall hear no more about it until Section 3. Since, on the other hand, we want '$X \geqslant Y$' to mean that X is a refinement (or 'further specification') of Y we shall require that $\geqslant$ is a weak partial ordering (whose strict companion, which we shall scarcely have occasion to refer to, we denote by '$>$'), and we insist, in addition, on the following two conditions:

Persistance: For any propositional variable π and any, $X, Y \in W$, if $V(\pi, X)$ is defined and $Y \geqslant X$, then $V(\pi, Y) = V(\pi, X)$.

Refinability: For any π, and any X, if $V(\pi, X)$ is undefined, then there exist Y and Z in W such that $Y \geqslant X$ with $V(\pi, Y) = T$ and $Z \geqslant X$ with $V(\pi, Z) = F$.

These conditions speak for themselves. Persistence is required because further delimitation of a possible state of affairs should not reverse truth-values, but only reduce indeterminacies, and Refinability says that (for atomic formulae at least) such a reduction is possible in either of the two relevant ways: this is a sort of 'principle of subdivision' for possibilities.[9] It should be noted that this principle does not require that any element of W should have a refinement in which the value of every propositional variable is defined.

It was with some trepidation that I adopted that '$\geqslant$' notation for refinement, since a case may be made out for an oppositely directed symbol ('$\leqslant$' or '$\subseteq$') here: for if one thinks of possible worlds diagrams (as in [18]) then its being the case that $X \geqslant Y$ corresponds in such a picture to the spatial region which represents X being included in the region representing Y (though of course, these relations may be thought of mereologically rather that set-theoretically). Nevertheless, the '$\geqslant$' appropriately captures the idea that, considered as a specification of what is the case, when $X \geqslant Y$, X extends Y.[10]

It remains to define the notion of truth for an arbitrary formula α at ('over', 'relative to', or whatever) an element X in a model $\langle W, \geqslant, R, V\rangle$, which we represent by: $\langle W, \geqslant, R, V\rangle, X \models \alpha$ (though the explicit reference to the model will be suppressed below). The general notion of validity of a formula will then be truth at every possibility in every model, and it is this characterization of validity that will, in Section 3, be shown to coincide with provability in the modal system K, and so to coincide with the usual notion of validity (i.e., truth in every model on any frame, with the domain of the frame consisting of possible worlds.)[11] The basis clause in the inductive definition of '$\models$' is obviously this, where, as before 'π' serves as a metalinguistic variable over propositional variables:

(Basis) $\quad X \models \pi \quad \text{iff} \quad V(\pi, X) = T.$

The inductive clauses for the non-modal connectives, and matters arising in connexion with them, will occupy us for the remainder of this section; modality is introduced in Section 3.

Let us take as our primitive non-modal connectives negation (written '$\sim$') and conjunction ('$\wedge$') with:

[Def. $\vee$] $\quad \alpha \vee \beta =_{\text{df}} \sim(\sim \alpha \wedge \sim \beta)$

and

[Def. $\supset$] $\quad \alpha \supset \beta =_{\text{df}} \sim(\alpha \wedge \sim \beta).$

We adopt these definitions because, wanting to end up with everything in classical truth-functional logic as valid on the present account, we want to preserve familiar interdefinabilities. Some remarks on the defined connectives will be made at the end of this section. For the moment, our problem is what to say about '$\sim$' and '$\wedge$'. What is said must be acceptable in the light of the informal account of Section 1, as well as resulting in our according validity to all (and, of course, only) those formulae in the present (i.e., '$\Box$'-free) language which are classically tautologous. The clause for conjunction in the definition of truth presents no problem; it should be this:

($\wedge$) $\quad X \models \alpha \wedge \beta \quad \text{iff} \quad X \models \alpha \quad \text{and} \quad X \models \beta.$

Surely a conjunction should be deemed to be true in a partially specified state of affairs just when each conjunct is true therein.[12] But a 'direct' clause for '$\sim$' parallelling this one would be inappropriate, since we want, as explained in Section 1, to regard the truth of the negation of a formula as recording the determinate falsity of the formula negated, and not just its failure to be true (which could arise because of underspecificity). Thus an intuitionist type of clause is called for:

$$(\sim) \qquad X \models \sim\alpha \quad \text{iff for all} \quad Y \geqslant X, Y \not\models \alpha.$$ [13]

We will now run through some obvious consequences of these definitions.

As in intuitionistic propositional logic, and for the same reasons (i.e., the proof is by induction on formula complexity, using Persistence for the basis case), we have:

LEMMA 1. If $Y \geqslant X$ and $X \models \alpha$, then $Y \models \alpha$, for any formula α.

And, recalling an early result of Gödel's ([14]), the intuitionistic parallels continue with:

LEMMA 2. For any formula α (of the present langauge), α is valid on our semantics iff α is a classical tautology.

Proof: 'Only if': if α is not a classical tautology, let the V of one of our models copy, on some possibility X, the assignments of truth-values to the variables of α which falsify it classically. 'If': Although a semantic argument could be given, it is easier to simply run through a set of axiom schemata in '$\sim$' and '$\wedge$' which is sufficient for classical propositional logic, and note that they are all valid, and that the rules preserve validity. (E.g., the axioms of [23], p. 306; to show modus ponens, in terms of '$\sim$' and '$\wedge$', preserves validity, Lemma 3 below may be helpful.)

Now, to show that two semantical conceptions validate the same set of formulae is not to show that they give rise to the same logic: for the semantic consequence relations engendered by the accounts may yet differ. And, notoriously, this is what happens in the case of the negation-conjunction fragments of intuitionistic and classical propositional logic. Thus, partly to reassure that nothing untoward in this way is going on, and partly because we need to know it for the completeness proof in Section 3, we should establish that a formula is true at a possibility iff its double negation is. Thus it is at this point that we part company with intuitionist logic, and the assumption of Refinability is exploited. The result appears in Lemma 4. First, a preliminary.

LEMMA 3. If $X \not\models \alpha$, then for some $Y \geqslant X$, $Y \models \sim \alpha$.[14]

Proof: By induction on formula complexity, using Refinability for the basis case.

LEMMA 4. $X \models \sim\sim \alpha$ iff $X \models \alpha$.

Proof: 'If': as in intuitionistic logic. 'Only if': By Lemma 3, one gets a contradiction from assuming that $X \models \sim\sim \alpha$ while $X \not\models \alpha$.

LEMMA 4 is given separately for convenience of reference; it is a special case of the following assurance that the consequence relation is entirely classical. (We write '$\alpha_1, \ldots, \alpha \mathrel{||\!\!-} \beta$' for: in any model, for any X in that model, *if* $X \models \alpha$; for $1 \leqslant i \leqslant n$, *then* $X \models \beta$.)

LEMMA 5. For any formulae $\alpha_1, \ldots, \alpha_n, \beta$: β is a tautological consequence of $\alpha_1, \ldots, \alpha_n$ iff $\alpha_1, \ldots, \alpha_n \mathrel{||\!\!-} \beta$.

Proof: 'If': As in the 'only if' part of the proof of Lemma 2. 'Only if': Suppose (1) that $\alpha_1, \ldots, \alpha_n$ have β as a tautological consequence, but (2) $\alpha_1, \ldots, \alpha_n \mathrel{||\!\!\not-} \beta$. (2) means that (in some model) there is an X such that $X \models \alpha_i$ for $1 \leqslant i \leqslant n$ while $X \not\models \beta$. Calling the conjunction of the α_i, 'α', then, by ($\wedge$), $X \models \alpha$. Since $X \not\models \beta$, by Lemma 3 there is a $Y \geqslant X$ with $Y \models \sim \beta$; and since $X \models \alpha$, by Lemma 1, $Y \models \alpha$; so $Y \models \alpha \wedge \sim \beta$. Now from (1) we have that $\sim(\alpha \wedge \sim \beta)$ is a tautology, and so true at every element in every model, by Lemma 2, and so that X, meaning that, since $Y \geqslant X$, $Y \models \alpha \wedge \sim \beta$, contradicting an earlier conclusion of ours. (It would do just as well to point out that this formula must be true at Y, since $Y \geqslant Y$.)

This official business out of the way, we close this section with a few remarks on the defined connectives '$\vee$' and '$\supset$'. First, a symbol-by-symbol unpacking of [Def. $\vee$] with the aid of ($\sim$) and ($\wedge$) gives disjunctions the following truth-conditions:

(∨) $X \models \alpha \vee \beta$ iff for all $Y \geqslant X$, either there exists $Z_1 \geqslant Y$ such that $Z_1 \models \alpha$, or there exists $Z_2 \geqslant Y$ such that $Z_2 \models \beta$.

Note that the right-hand side here simplifies to: for all $Y \geqslant X$, there is a $Z \geqslant Y$ such that either $Z \models \alpha$ or $Z \models \beta$. Not very simple, all the same. But that is just how it is for possibilities as opposed to worlds: the fact that a 'simple' truth condition, analogous to (∧), is not available reflects the fact that a disjunction can be true at a possibility without either disjunct's being true there – the hallmark of indeterminacy, one might say. (One could use a direct clause for '∨' and add a determinacy operator, 'D', say,[15] then distinguishing between $D(\alpha \vee \beta)$ and $D\alpha \vee D\beta$, but this would be to go against the general procedural constraint mentioned in Section 1, since we would then be tampering with the object language.[16])

Let us turn to implication. Unpacking [Def. ⊃] with the aid of (~) and (∧) yields:

(⊃) $X \models \alpha \supset \beta$ iff for all $Y \geqslant X$, if $Y \models \alpha$ then there exists $Z \geqslant Y$ such that $Z \models \beta$.[17]

This somewhat unusual clause may be compared with a direct truth condition we may imagine a connective '→' to be equipped with. ('→' is not a permanent addition to our language, and is introduced here only for comparative purposes.) That is:

(→) $X \models \alpha \rightarrow \beta$ iff either $X \not\models \alpha$ or $X \models \beta$,

It is instructive to compare the relative strengths of the conditionals $\alpha \supset \beta$ and $\alpha \rightarrow \beta$, for →-free α and β. The symbolization is actually rather misleading, in that it is the former conditional which is the stronger. To see that $\alpha \supset \beta \Vdash \alpha \rightarrow \beta$, suppose that $X \models \alpha \supset \beta$ while $X \not\models \alpha \rightarrow \beta$. This means $X \models \alpha$ and $X \not\models \beta$; so, by Lemma 3, there is a $Y \geqslant X$ with $Y \models \sim\beta$. But by Lemma 1, $Y \models \alpha$, which contradicts the assumption that $X \models \alpha \supset \beta$, since Y is a refinement of X no further refinement of which will verify β. (Note that Lemmas 1 and 3 are guaranteed to apply here only because we stipulated that α and β were to be in our official language, excluding occurrences of '→'.) On the other hand, it is not the case that $\alpha \rightarrow \beta \Vdash \alpha \supset \beta$. For a counterexample, choose for α a propositional variable whose value (under V) is undefined with respect to a possibility X (in some model), and for β a propositional variable such that $V(\beta, Y) = F$ for all Y (in the model); then

$X \models \alpha \rightarrow \beta$ while $X \not\models \alpha \supset \beta$. As this example brings out, '$\rightarrow$' is a rather undesirable connective, in that it introduces failures of persistence (at least, if I am right about the undesirability of this, in note 16); indeed the classical style of negation mentioned in note 16 as the fountainhead of such failures is definable in terms of '$\rightarrow$' since $X \not\models \gamma$ iff $X \models \gamma \rightarrow (\gamma \wedge \sim \gamma)$, for any γ. Although $\alpha \supset \beta$ is not a consequence of $\alpha \rightarrow \beta$ in the sense of being true at any possibility in any model at which $\alpha \rightarrow \beta$ is true, it *is* the case that if $\alpha \rightarrow \beta$ is a valid formula, then so is $\alpha \supset \beta$.

Next, we may compare '$\supset$' with intuitionistic implication, for which we coin – again, for temporary use only – the symbol '$\Rightarrow$':

($\Rightarrow$) $\quad X \models \alpha \Rightarrow \beta \quad$ iff for all $Y \geqslant X$, if $Y \models \alpha$, then $Y \models \beta$.

Not only does '$\Rightarrow$', unlike '$\rightarrow$', always form presistent compounds from persistent components, so that Lemma 1 hold for our language extended by '$\Rightarrow$' (familiar from intuitionistic logic); but the new inductive case it presents for the proof of Lemma 3 also goes through. This means that in comparing $\alpha \supset \beta$ with $\alpha \Rightarrow \beta$, we will need to place no restrictions on the composition of α and β. In fact, this comparison yields the convenient conclusion that these two conditionals are equivalent in the strongest possible sense. That $\alpha \Rightarrow \beta \mid\vdash \alpha \supset \beta$ follows from the reflexivity of $\geqslant$. And, conversely, suppose that $X \not\models \alpha \Rightarrow \beta$, so that for some $Y \geqslant X$, $Y \models \alpha$ while $Y \not\models \beta$. Then (by Lemma 3), for some $Z \geqslant Y, Z \models \sim \beta$. But $Z \models \alpha$ (Lemma 1). Clearly Z is a refinement of X (by transitivity of $\geqslant$) at which α is true, but at no further refinement of which is β true, so $X \not\models \alpha \supset \beta$. I describe this result as convenient because it means that we can employ the more simply formulated right-hand side of ($\Rightarrow$) instead of that of ($\supset$), whenever we are working in a language in which Lemmas 1 and 3 continue to hold.

3. MODALITY

Let us add to the language of Section 2 the modal operator '$\Box$' ('it is necessarily the case that') interpreted, predictably, by universal quantification over 'accessible' possibilities, i.e., with the following clause:

($\Box$) $\quad \langle W, \geqslant, R, V\rangle\, X \models \Box\alpha \quad$ iff for all $Y \in W$ such that RXY, $\langle W, \geqslant, R, V\rangle\, Y \models \alpha$.

As in the previous section, reference to the model will usually be kept tacit.

Further, we impose, for the purpose of interpreting this modal language, certain extra conditions on the structures $\langle W, \geqslant, R, V \rangle$ of Section 2 for these to count as models. These conditions concern the interconnexions between the two relations $\geqslant$ and R. One does not want to allow arbitrarily related refinement and accessibility relations, in view of the intuitive ideas these relations are used to capture. In the first place, justice must be done to the idea that refinements merely refine (i.e., render determinate): they do not turn what was true into something no longer true. This is why Lemma 1 of Section 2 ought to be secured for our new language with '$\Box$'. For the inductive case involving '$\Box$' we need to be assured that, at least when attached to a persistent formula, this operator yields a formula which is persistent. This is guaranteed by the following condition on models:

(P1) for all X, X' and Y, if $X' \geqslant X$ and $RX'Y$, then RXY.

Though less vital for present purposes than it would be in tense logic (because '$\Box$' does not have a 'converse' operator), let us record also the twin condition:

(P2) For all X, Y and Y', if $Y' \geqslant Y$ and RXY, then RXY'.

The third requirement we impose is best stated with the aid of an abbreviation: we write 'R^+XY' for 'for all $X' \geqslant X, RX'\,Y$'. The effect of this condition will be to guarantee generalized refinability (i.e., the analogue of Lemma 3):

(R) For all X, Y, if RXY then for some $X' \geqslant X, R^+X'Y$.

Let us briefly consider the plausibility of these conditons. Having already explained how we are to think of the relation $\geqslant$, a word is in order on the relation R. I am thinking of its being the case that RXY along the following lines: whatever is determined in the possibility X as necessarily true is determined in Y is true. ('Determined as true' simply means 'determinately true', and thus is just a thetorical variant on 'true'.) With this construal in mind, let us look at **(P1)**. For its antecedent to hold and its consequent to fail something necessary in X would need to be not true in Y. But the necessitated formula in question would have to be true in X', since it is true in X and nothing every ceases to be true on passing from a possibility to a refinement thereof (and $X' \geqslant X$), which contradicts the assumption that $RX'Y$, on the present construal of R. The case for **(P2)** is similar. When we turn to

(R), the first question to raise is why we do not require -more simply- that for any X and Y if RXY then R^+XY. In other words, do we really wish to allow that we might have $RXY, X' \geqslant X$, yet not $RX'Y$? The answer is that we certainly do wish to allow this. For let it be, at X, determinately true that either $\Box\alpha$ or $\Box\sim\alpha$, without either disjunct's being determinately true here. Compatibly with this, we could certainly have a possibility Y at which α is true being R-accessible to X (as long as Y verifies anything else whose necessitation is true at X). But now X has refinements X_1 at which $\Box\alpha$ is true and X_2, at which $\Box\sim\alpha$ is true, and while X_1 may well bear R to Y, X_2 clearly does not. Hence the unacceptability of the envisaged simplification of **(R)**. But what of **(R)** itself? Here we argue that when RXY, even if it is not X itself that bears R^+ to Y, we can always find some $X' \geqslant X$ such that R^+XY, because such an X' is described by extending the description of (i.e., set of formulae true at) X with the set of all formulae $\sim\Box\beta$ for which β is not true at Y.

From now on, I shall take the conditions **(P1)**, **(P2)** and **(R)** as being in force whenever we are talking of models. The reader will be able to use them (in fact, **(P1)** and **(R)**) to reconstruct the inductive steps for '$\Box$' in the proofs of Lemmas 1 and 3 of Section 2, so that all of the results of that section may now be taken to apply to the modal language. (Lemmas 2 and 5 being understood so that the notions of tautology and tautological consequence apply – by substitution – to formulae continaing '$\Box$'.)

Recall that the modal system K may be presented as having for its axioms:

(1) All substitution instances in the present language of truth-functional tautologies.

and

(2) All formulae of the form $\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$

and for its rules of proof, *modus ponens* and necessitation. A basic result in modal logic is that provability in this system coincides with truth at every world in every model, i.e., with the most general conception of validity available in the Kripke ('relational') semantics. The main item on the agenda for this section will be to show that provability in K also coincides with the most general conception of validity formulable within the present framework: truth with respect to every possibility in every model on *our* sense. We want to show –

THEOREM: A formula is valid iff it is provable in K.

Proof: 'Only if' (i.e., soundness). Axioms falling under (1) are all valid, by Lemma 2. As for (2), suppose that for some model $\langle W, \geqslant, R, V \rangle$ and $X \in W$, $X \not\models (2)$. Using Lemmas 1 and 3, there is an $X' \geqslant X$ such that $X' \models \Box(\alpha \supset \beta)$ and $X' \models \Box\alpha$ and $X' \not\models \Box\beta$. By ($\Box$) there is a Y such that $X'RY$ and $Y \not\models \beta$. But also by ($\Box$), $Y \models \alpha \supset \beta$ and $Y \models \alpha$, which is impossible. Notice that this is just the usual reasoning.[18] Also for the usual reasons, necessitation and modus ponens preserve validity.[19]

'If' (completeness). It suffices to note that an arbitrary Kripke model $\langle W, R, V \rangle$ can be converted onto an equivalent model in our sense, $\langle W, \geqslant, R, V \rangle$ by letting $X \geqslant Y$ iff $X = Y$.[20] (Note that **(P1)**, **(P2)** and **(R)** are satisfied.)

The completeness proof given here trades on the known completeness of K with respect to the class of all Kripke models. In an earlier version of this paper, I offered a variant of the canonical model proof of the latter fact in order to establish the former result 'directly', but as the key device used in that construction has since appeared in Röper [24], I shall omit the details here, mentioning only that the device in question is that of taking not maximal K-consistent sets of formulae, as in the usual proof, but instead sets of formulae which are both consistent and 'dedictively closed' relative to K. With the usual definitions of the R and V of the canonical model, and taking $X \geqslant Y$ to mean that $Y \subseteq X$, it can be shown that for any formula α, and any X from this canonical model $X \models \alpha$ iff $\alpha \in X$ (iff, we may add, $X \Vdash \alpha$), which gives a completeness proof arguably of greater interest on two counts. The first is that such a proof may appeal to someone of strongly anti-infinitistic temperament, to whom some of the qualms about possible worlds aired in [1] might be especially forceful, as avoiding the infinite union step which figures in the proof of Lindenbaum's Lemma required for the standard construction. To be sure, the alternative proof involves infinite sets of formulae, but *need* only involve such as are 'finitely generated' (i.e., as consist of the consequences of some finite set). This is somewhat like wishing to reason about intuitionistic logic in intuitionistically acceptable ways. The second point of interest in the alternative method of proof is that it lends itself to proofs of completeness where the collapsing of '$\geqslant$' into '$=$' won't work. For example, K can be shown to be complete with respect to the class of models meeting the 'infinite descent' conditon mentioned in note 9:

For all X there exists a Y such that $Y > X$.

The proof makes essential use of finitely generated (consistent and deductively closed) sets, and so does not yield (as the proof of the Theorem above does) *strong* completeness for K with respect to the class of models indicated.

Completeness for those familiar extensions of K which are axiomatized by schemata of the form $0\,\alpha \supset 0'\alpha$, where 0 and $0'$ are (possibly empty) sequences of occurrences of '□' may be shown by the standard arguments. For example, with $0 =$ '□', and $0'$ empty, we have (soundless and) completeness with respect to the class of those models in which R is reflexive; and with $0 =$ '□' and $0' =$ '□□', the relevant class consists of those models in which R is transitive. (These are the systems normally called T and K4, respectively; the results are, as usual 'additive'.) When we come to axioms which interlace occurrences of '∼' with those of '□', the usual arguments do *not*, however, apply. A representative (and intrinsically important) case will be treated at the end of this section.

As our final topic for this section, let us have a look at possibility ('◇') as defined connective:

[Def. ◇] $\quad \Diamond\alpha =_{\mathrm{df}} \sim\Box\sim\alpha.$

This gives the following truth-conditions:

(◇) $\quad X \models \Diamond\alpha$ iff for all $X' \geqslant X$ there is a Y such that $RX'Y$ and there is a $Y' \geqslant Y$ such that $Y' \models \alpha$.

Now this certainly seems like quite a mouthful when what one is expecting is a clause analogous to (□) but with 'some' in place of 'all' on the right-hand side. To facilitate a comparative discussion of the truth-conditions given for '◇' and those might have been expected, let us follow the practice of Section 2 and (temporarily) introduce a modal operator '*M*' with the expected truth-conditions, and contrast it with '◇':

(*M*) $\quad X \models M\alpha$ iff for some Y such that RXY, $Y \models \alpha$.

When we compare the relative strength of the two possibility operators, we find that $\Diamond\alpha \mid\vdash M\alpha$, but not conversely. To see that $\Diamond\alpha \mid\vdash M\alpha$, suppose that $X \models \Diamond\alpha$; then looking at ($\Diamond$)) every refinement of X has access to some element with a refinement verifying α. So X, being a refinement of itself, has access to (i.e., bears R to) some Y for which there exists a $Y' > Y$ such that $Y' \models \alpha$. A picture may help. The horizontal arrow goes from a point to a point to which it bears R; the vertical arrow goes from a possibility to a refinement thereof:

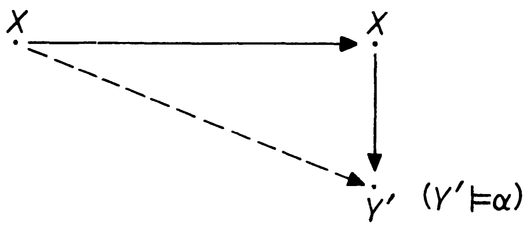

Fig. 1

The broken line, indicating the relation R – think of it as more horizontal than vertical – may now be added, by (**P2**), so that $X \models M\alpha$, as was to be shown.

The most instructive way to see that, on the other hand, '$X \models M\alpha$' does not imply '$X \models \Diamond\alpha$' is via the observation that formulae of the form $M\alpha$ are not (in general) persistent, and so cannot be equivalent to formulae of the form $\Diamond\alpha$, which are, being abbreviated versions of formulae in our official language, persistent. Why are such formulae not persistent? For the truth of $M\alpha$ at X to imply the truth of $M\alpha$ at $X' \geqslant X$, we should have to be assured that X' bore R to any element to which X bore R to; in other words, we should have to be able to complete, with the broken arrow shown here, the following diagram:

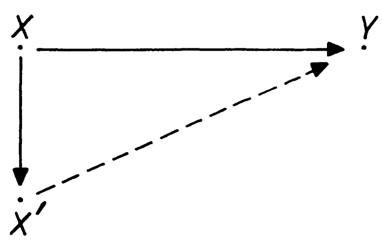

Fig. 2.

But we have already seen, in discussing the simplifiability of the condition **(R)** when it was first introduced, that such a diagram cannot in general be completed in this way.[21]

It is in fact worth commenting on that the principle we have just attended to as objectionable, which would allow us to insert the north-easterly arrow in Figure 2, would follow from what we are committed to if we imposed, in addition, the further condition that the relation R should be symmetric. For suppose (1) RXY and (2) $X' \geqslant X$. Then, from (1) by symmetry, it follows that RYX, and from this and (2), we infer, by **(P1)**, that RYX', so again by symmetry, we conclude that $RX'Y$. It looks, then, as though if our objections to this principle are correct, we must avoid restricting our attention to the class of symmetric models, or any subclass thereof. But this seems initially something of an embarrassment, since it threatens to debar us from considering any extensions of the system KB axiomatized by adding to any set of axioms for K, instances of the following schema:

(B) $\alpha \supset \Box \Diamond \alpha.$

For it is precisely this system which, in the usual 'worlds' framework, is sound and complete with respect to the class of all models whose accessibility relations are symmetric. And this *would* be embarrassing, since it would put S5, the most widely favoured modal logic from the point of view of philosophical applications, forever outside the reach of our 'possibilities' framework. However, there is, as far as I can see, no cause for despair on any such grounds. For there is a condition we can impose on models which does not imply symmetry in general, but which does delimit a class of models such that truth at every element of these models, coincides with provability in KB, namely:

(*) If R^+XY and $Y' > Y$, then $RY'X$.

Let us check soundness first. For our axiom (B) to fail to be true at some element W in a model meeting (*), we must have some $X \geqslant W$ with $X \models \alpha$ while $X \not\models \Box \Diamond \alpha$. So there exists Y such that RXY and $Y \not\models \Diamond \alpha$. And, further, by **(R)** there is some refinement X' of X such that $R^+X'Y$. The situation described so far is as pictured in Figure 3, with formulae exhibited in primitive notation:

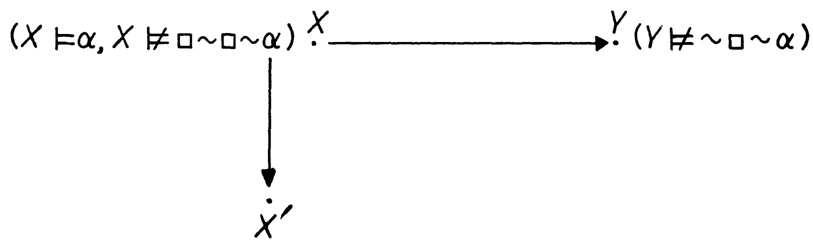

Fig. 3.

Since $Y \not\models \sim \Box \sim \alpha$, there is, by Lemmas 3 and 4, as $Y' \geqslant Y$ with $Y' \models \Box \sim \alpha$. Then, since $R^{+}X'Y$ and $Y' > Y$, (*) tells that $RY'X'$. (The reader may care to copy Figure 3, inserting this new point Y' in the lower right-hand corner, with an arrow going from it to X'.) Now we have a contradiction, for, being a refinement of X, at which α is true, X' must verify α, by Lemma 1, but, being accessible to Y', at which $\Box \sim \alpha$ is true, X' must verify $\sim \alpha$: and X' cannot do both.

We see, that, the *KB* is sound and complete with respect to the class of models satisfying condition (*). By composing arguments that we have either given or alluded to, one has the result that S5 is similarly determined by the class of models meeting (*) in which, in addition, the relation R is both reflexive and transitive. (A simpler approach to the semantics of S5 in the possibilities framework, analogous to that in the worlds framework which does not mention the relation R, is mentioned in Subsection A of Section 4.) The infinite descent condition, mentioned above, to the effect that every element has a proper refinement, may be thrown in too without jeopardy to this completeness result, as the result can also be shown 'directly' by what I earlier called the alternative method of deductively closed sets, instead of going via a standard completeness theorem. This concludes our case for the claim that, if one is prepared to put up with those complexities which are forced on us by the deviant treatment negation receives in the possibilities framework, this framework provides a setting for the model-theoretic semantics of modal logic no less usable in that capacity than the possible worlds framework customarily assumed.

4. RESIDUA

There are several topics on which further work would be desirable, arising out of the above discussion. For example, the results of Section 3 might be extended to other systems of modal logic. Further, the orientation of that section was selected somewhat arbitrarily from various alternatives, which might usefully be explored – for example, that mentioned in note 21: or there might be a satisfactory way of treating possibility and necessity as non-interdefinable. Again, on the subject of the definition of truth, the relative merits of the above approach and one in the style of Röper [24] might be compared.[22] On the latter, one defines $\Box\,\alpha$ to be true at X iff for all $X' \geqslant X$ and all Y such that $RX'Y$, α is true at Y.[22] Such a treatment builds persistence into $\Box$-formulae by the definition (as($\sim$) of Section 2 does, but ($\wedge$) does not) rather than extracting it as a consequence of a condition on models. Finally, one could consider various extensions of the language of Section 3, of which the most important would be the addition of quantifiers and variables. Here various nice questions arise about what it is reasonable to demand for the truth of a universally quantified formula at a possibility. (My preferred answer to this one is that the truth-condition should involve quantification not only over such objects as belong to the domain of the possibility in question, but over those inhabiting any of its refinements.) There may also be some interest in applying the above framework to the language of (counterfactual or other) conditionals, in particular because even if both what Lewis [18] calls the Limit Assumption and what he calls Stalnaker's assumption are imposed, understood with reference to possibilities instead of worlds, there remains a distinction between the failure of β to be true at the closest α-verifying possibility to X, and $\sim\beta$'s being true thereat. This section will now close the paper by giving slightly fuller discussion to two topics not included in the above list, for which the possibilities framework promises a distinctive treatment.

A. *Propositions*

Where $\mathscr{M}$ is a model of the usual sort in the possible worlds semantics for modal logic, it has become customary to think of the proposition expressed by a formula α relative to $\mathscr{M}$ as simply α's truth-set (i.e., in the usual notation $\{w\colon \mathscr{M} \models_w \alpha\}$). Realists about possible worlds, possessed of the conception of an intended model, and others too, as if possessed of such a conception, then take the proposition expressed by a sentence of a natural language to be the truth-set of that sentence in the intended model. I think it is pretty widely conceded that this is, at least, a very convenient way of construing talk of propositions, when it is logical relations between (and operations upon) propositions that are at issue, since these then emerge as familiar set-theoretic relations (and operations), even if this truth-set conceptualization lets us down when we come to want propositions as the objects of the propositional attitudes. What would be an analogously suitable way of representing propositions on the present framework? The first answer that comes to mind is: again, use the appropriate truth-sets. This time they will consist not of worlds, but of possibilities, the proposition expressed by α relative to a model $\mathscr{M}$ (this being a model on the sense of Sections 2 or 3 of this paper) being $\{X: \mathscr{M}\ X \models \alpha\}$. Then the proposition expressed by a conjunction is in the intersection of the propositions expressed by the conjuncts, as on the usual approach, while the earlier clause ($\sim$) gives something rather cumbersome by comparison with the neat complementation principle available on the ususal approach. (The case of the proposition expressed by a necessitated formula is equally cumbersome on both accounts.)

A second answer to the question of how to represent propositions on the possibilities, as opposed to the possible worlds, approach, would be to take them not as sets of possibilities, but simply as possibilities (roughly). Why not take the proposition expressed by α to be the region of logical space over which α is true (i.e., the largest such region)? I add the parenthetical 'roughly' because if we want every formula (or sentence) to express a proposition, then (e.g.) '$p \wedge \sim p$' will express something which only with considerable gritting of teeth could be called a possibility. I would rather use 'region' as the generic term here, regions comprising *bona fide* possibilities together with the empty region. The set of regions is to be fitted out with appropriate mereological relations of intersection, union (usually called 'fusion' in this context) and complementation, as in the case of the part-whole mereology of spatial regions or physical objects.[23] In the latter connexion, the empty object is not usually recognized, presumably because such mereological developments (the 'calculus of individuals', etc.) have been the work especially of nominalistically minded philosophers who have misgivings (which they call 'scruples') about its abstractness; but there seems no reason to exclude it here, thus letting the regions with the operations just mentioned determine an atomless boolean algebra. (Taking just the possibilities we don't get this, there being no zero.) Atomlessness here corresponds to what was called the condition of infinite descent above. (That condition must now be read with care. If our variables range over regions, rather than, as before, possibilities, we must say that for every non-empty X, there is a non-empty Y such that Y is a proper refinement of X, the refinement relation being interdefinable with the boolean operations in an obvious way.) Representing by '$[[\alpha]]$' the proposition expressed by α, in accordance with this suggestion, we have that α is true over/at X iff $X \geqslant [[\alpha]]$, $[[\alpha \wedge \beta]]$ is $[[\alpha]] \cap [[\beta]]$, and – this time a simple principle for negation – $[[\sim\alpha]]$ is the complement of $[[\alpha]]$: the union, or fusion, of all those region which do not overlap (i.e., do not have a non-empty intersection with) $[[\alpha]]$. I think this second way of construing propositions may be worth investigating. The following subsection looks at one possible application.

B. *Beliefs*

Finally, I should like to offer a few words on the use of possibilities as the objects of (consistent) belief.[24] In thinking of belief from a logical point of view, it is convenient to distinguish between two batches of idealizing assumptions which get canonized into axioms for doxastic logic, and may be associated with various conditions on the appropriate models in the possible worlds framework for such studies. In the first batch there are what might be called *rationality assumptions*, which, in what I hope is self-explanatory terminology are (i) logical omniscience, (ii) deductive closure, and (iii) consistency. This gets quite a bit of doxastic logic going (giving the 'belief' operator the logic KD), but even more idealisation is called for if we want the simplest possible story, so we may be led to consider as a second batch, a pair of self-awareness assumptions, (iv) to the effect that if the subject believes that something is the case, he believes that he believes this, and (v) if he does not believe something, then he believes that he does not believe it: in other words, the S4 and S5 axioms for the operator in question. Then models for the resulting logic (KDE4 in the terminology of [17]) may be taken to consist of a set of worlds together with a distinguished non-empty subset thereof, and a valuation: the worlds in the special subset are just the belief-compatible worlds. (Validity is truth at *every* world in every model.) It is from this grossly simplified point of view that I will comment on the situation.

The philosophical man in the street occasionally speaks of so-and-so's 'belief-world': the world as so-and-so believes it to be. And philosophers rightly protest at such a terminology since even with all the idealizing assumptions on (including, especially, consistency) a person may believe a disjunction without believing either disjunct, so that there really is no such thing as *the* world as so-and-so believes it to be. Rather, in such cases as the one just envisaged, what we have is that the disjunction holds at every one of so-and-so's belief-compatible worlds, the one disjunct holding at some, the other at others. Being believed is a matter of holding at all such worlds, and the indeterminacy just noted is nothing but the familiar failure of the universal quantifier to distribute over disjunction. The philosophers' objection to the 'belief-world' locution is well taken, and it may be respected while at the same time justice is done to the man in the street's idea that (not quite putting it in his own terms) there should be a single entity such that truth with respect to if coincides with being believed by so-and-so: the entity won't be a possible world, but, instead, a possibility. There is nothing surprising in this, of couse, since the whole point of the transition to possibilities is to escape unwanted determinacy.[25] (This is probably the upshot of Tennant's suggestion in [27], disregarding his illadvised talk of 'three-valued models' (p. 427). I say 'probably' because [27] is only a brief informal summary of Tennant's account.) Models in the possible worlds framework just described as suitable for doxastic logic with both batches of assumptions in force go over into the possibilities framework as models in the sense of this paper minus the accessibility relation and with a distinguished possibility – call it '$\underline{B}$' – so that, writing 'B' for the doxastic operator, we have, for any X, relative to such a model, $X \models B\alpha$ iff $\underline{B} \models \alpha$. Then a semantic treatment may be given along the lines of Section 3, in which fully determinate doxastic alternatives never enter the picture. Indeed, we may think of any possibility as a candidate for the content of someone's corpus of beliefs – assuming consistency – and this may be of interest to those in sympathy with Brian Ellis' attempt (see [10], [11]) to banish truth from semantics altogether, pursuing the subject instead exclusively in terms of incorporability into 'rational belief systems'. (Read '$X \models \alpha$' as, 'According to belief-system X, α holds', or, even 'X believes that α'.) Whatever the merits of this programme (the obvious difficulty being whether we can really make sense of the rationality of a set of beliefs other than in terms of the possibility of their joint truth), something that has always struck me as a distasteful feature of its execution is Ellis's use of completed belief-systems, since these are fully determinate sets of beliefs and are thus at a considerable remove from any actual belief-systems. But the idealization of complete-ability, as Ellis calls it, is not in fact required. As Sections 2 and 3 of this paper show, all one needs for classical logic to flow from a semantic framework admitting indeterminacies is that the indeterminate entities with respect to which formulae hold and fail to hold should be capable of having any *given* indeterminacy resolved. We need that these elements can always be capable of being made more determinate ('refined' as I say): not that any of them should be made *completely* determinate.

It is worth remarking that modality itself may be given a treatment along the lines suggested for belief in the previous paragraph, which affords an account at once more natural but less general than that of Section 3. (The present story works, as far as I can see, only for S5.) The idea is that, denoting the fusion of W by 'UW', we have as models $\langle W, \geqslant, V\rangle$ – as before only minus the relation R – with the following clause for '$\Box$', where $\mathcal{M}$ is any such model and $X \in W$: $\mathcal{M} X \models \Box\alpha$ iff $\mathcal{M}, UW \models \alpha$. What is necessarily true is what holds even over the completely unspecified possibility. (Better still: let the 'W' of our models denote not a set of regions but a region, of which the other regions in the model are refinements, so that the 'UW' of the clause just given may be replaced by 'W'.) If validity is defined as truth with respect to every possibility in every such model, then validity can be shown to coincide with provability in the modal system S5. Further, possibility, '$\Diamond$' defined as in Section 3, turns out to have an attractive effect on truth-conditions: $\mathcal{M}, X \models \Diamond\alpha$ iff for some $Y \in W, \mathcal{M}, Y \models \alpha$, so that there isn't the awkard divergence between the '$\Diamond$' and 'M' of that section.

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NOTES

[1] I say I 'attempted' to do tense logic with intervals in place of instants in [15], because Section 4 of that paper, the only section concerned with tense logic proper, is marred by a serious error in the completeness proof for the basic system. At the base of p. 186 I say that any formula implies its own double intuitionistic negation: and this is incorrect (holding only for the formulae there described as persistent). This mistake was pointed out to me by Professor R. F. Barnes, and will be corrected when his current work on intervals is published. For references to other logicians who have been working in the same area, see van Benthem [3], as well as note 8, below.

[2] See [15], note 3, for a brief discussion of this idea, and also for reference to a discussion of the idea in the following sentence.

[3] Usually this complaint is lumped together with a similar objection to saying that there *is* a possible individual which is F on the basis of its being possible that there should be an individual which is F. In fact, these two objections should be separated since some difficulties afflicting realism about possible individuals (e.g., identity problems) do not touch possible worlds. (I learnt this point from M. K. Davies [6], p. 174 f.) For reasons of space, I am not exploring the variety of conceptions of possible worlds on which philosophers have considered themselves to be realists.

[4] This use of 'logical space' is not the only one current; for example, for van Fraassen (e.g., [13]) a logical space is something like a range of properties and a point therein is something like a slot defined in terms of these properties which may or may not be occupied by an individual.

[5] From [18], p. 84. In [20], at p. 533, Lewis partly retracts this argument, because of the observation by R. Stalnaker that it makes possible worlds abstract things ('ways things might have have been') and so does not fit in well with the main tenor of [18], in which worlds are seen as concrete particulars.

[6] The remarks appear at p. 57 of [5]. The theme is picked up in [7] and developed in a somewhat different way from my own. It is perhaps also worth mentioning here that the main scientific use of possible worlds, i.e., as the set of 'outcomes' or 'events' in probability theory, does not require the fully determinate worlds of current metaphysics and philosophical logic, but only what we are calling possibilities. See especially the discussion of 'small worlds' in Savage [25]; more generally, the modal undercurrents of probability theory and statistics are discussed in Suppes [26].

[7] See for example [8]. A related, though rather more traditional, view is the doctrine of the determinacy of the real (as exemplified in, for instance, D. M. Armstrong's argument in [1], p. 220, against sense-data on the grounds that a sense-datum would have to be a not fully determinate entity). This gives a separate motivation – unless one is prepared, with Lewis, to make a distinction between 'real' and 'actual' – for denying that there are non-actual possible worlds while agreeing that there are unactualized possibilities.

[8] Even greater would be the similarity to the treatment of interval based tense logic in the very instructive paper [24] by Peter Röper, which came to my attention after the work reported here was completed.

[9] Both conditions may be found in Fine [12]; it is the second condition which keeps the logic classical in spite of the intuitionist-style truth-conditions for negated formulae given below. The condition of Refinability allows for infinite 'descending' sequences (or 'ascending', if one thinks of the choice of the symbol '$\geqslant$') in respect of further specification of possibilities. This was mentioned in Section 1 as a possible condition to impose on models in order to push the motivation ideas of that section more aggressively and *require*, rather than merely *allow*, possibilities to be indeterminate. This condition, analogous to the condition (Subint) of [15], would read: for every X there is a Y such that $Y > X$. As will be explained below (Section 3), imposing this condition on models makes no difference to the class of valid formulae (in the language investigated in this paper).

[10] The term 'extends' is used in this sort of way by Fine, who also (in [12]) used, in addition to the '$\models$' of the next paragraph, a metalinguistic falsity predicate ('$\Dashv$') where I make do with the truth of negations (and with V assigning F, in the atomic case).

[11] It might be suggested that we define a valid formula as one which is false at no possibility in any model, rather than true at every possibility, in order to smooth the way here. But validity is intended as the precise correlate of the intuitive idea of logical truth, and a logical truth is surely something which is true relative to any possibility, which is why I prefer the present definition of validity (though, as it happens, under the assignment of truth-conditions adopted in this section, the two conceptions of validity coincide in extension).

[12] For stylistic variation, I allow myself here a form of words which might be taken to be quite misleading from the metaphysical perspective of Section 1: referring to a possibility as a partially specified state of affairs conveys the suggestion that we are operating with incomplete or partial descriptions of the traditionally determinate possible worlds.

[13] '$Y \mid\not\models \alpha$' abbreviates: it is not the case that $Y \models \alpha$; this metalinguistic negation is understood 'classically' (i.e., bivalence reigns). Similarly with the notation '$\mid\not\models$' below.

[14] We could state the consequent thus: for some $Y \geqslant X$, there is no $Z \geqslant Y$ such that $Z \models \alpha$. This Lemma may be regarded as a sort of generalization of Refinability to arbitrary formulae (similarly with Lemma 1 and Persistence).

[15] 'D' would be analogous to the combination '$\neg\sim$' of [15]; its semantics would be given by universal quantification over refinements.

[16] Similarly, using 'D', one could get by with a negation with a classical semantic clause (writing our own negation then as 'D' followed by that connective); but this goes against the spirit of the present enterprise since it would give rise to formulae which were not persistent into refinements (i.e., for which Lemma 1 did not hold), and this undermines the idea of refinements are mere resolvers of indeterminacies.

[17] Metalinguistic conditionals are to be understood materially, as usual. In the clause for '$\rightarrow$' below, I have used the disjunctive wording for reasons of euphony.

[18] Here, as well as at some other points in this paper, I incorporate a correction due to the referee of this journal.

[19] These results in fact preserve the property of being-true-at-every-element for any given model, and *a fortiori*, preserves validity.

[20] Taking the refinement relation as identity collapses our semantic clauses for the connectives into those of the standard Kripke semantics. The equivalence of models is then a point-by-point agreement on the truth-values of all formulae.

[21] It should be noted that the impersistence of M-formulae, as understood here, reflects our initial bias in relating '$\Box$' and the relation R as we did in explaining how R was to be thought of. An alternative construal of R could have started with M-formulae and run: RXY when any formula true in Y is possible (in the sense of 'M') in X, with notions of persistence and refinability tailored to match.

[22] Röper's approach may indeed be better than that of [15] in tense logic, for the reasons he gives at p. 454.

[23] Some details, as well as references to the relevant literature, may be found in [9]; see also [15], Section 3, and the references there to Hamblin's work. (The tense-logical case is somewhat different in that arbitrary unions of intervals need not be intervals. I believe that it might be possible to distinguish scattered from non-scattered individuals in the modal case too, by appeal to inter-world similarity, as in Bigelow [4], and that this may be of assistance in giving a non-epistemological account of a simple proposition and a genuinely disjunctive proposition, as these figure in Pollock [22]. I will not develop the suggestion here as it is independent of the worlds-*vs*.-possibilities issue.)

[24] Inconsistency does not seem well treated by the assignment of the empty region as the content of belief, or by somehow manufacturing a plurality of such regions (any more than by the introduction of impossible worlds in the worlds framework). A more promising line seems to be that of Lewis, who, in [19], investigates the idea of dividing up a person's beliefs into internally consistent, even if mutually inconsistent, fragments, which are then more tractable on either the worlds or the possibilities approach.

[25] A parallel indeterminacy arising in the case of perception is one of the considerations which leads Barwise in [2] to conclude that no modal style of treatment can be right for a certain ('bare infinitive') perceptual verb construction. I cannot here enter into a discussion of the account Barwise favours, however.

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