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A BIT OF A NOTE ON HALTING PROBABILITIES

We take a look at the definition of a halting probability from logical and algorithmic information theoretical points of view.

Keywords: algorithm, halting probabilities, information.

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Предложено рассмотрение определения вероятности остановки с точки зрения логической и алгоритмической теории информации.

Ключевые слова: алгоритм, вероятности остановки, информация.

Introduction

A coin is said to be *fair*, when the probability of getting a head (H) by tossing it is equal to the probability of getting a tail (T). Thus, each probability is 1/2 since they should add up to 1 by Kolmogorov's axioms of probability measure. So, the probability of getting a first tail by tossing a fair coin two times (that is, getting either TH or TT) is again 1/2. Let us toss a fair coin one or two times and ask what the probability of getting either one head (that is, H) or two sides that begin with a tail (that is, TH or TT) is. To choose the number of tosses, we provide an urn containing two similar balls, each with a unique label of 1 or 2. We close our eyes and pick up a ball from the urn. If the ball is labeled 1, then we toss the fair coin once; if it is labeled 2, then we toss twice. What is the probability that we get H, TH, or TT?

Whatever that is, let us consider the complement of this event. What is the probability of getting T, HT, or HH by tossing a fair coin once or twice randomly? The probability of H should be equal to the probability of T; call their common value q. This is the definition of a fair coin. So should be the probabilities of TH, TT, HT, and HH; call their common value r. So, the probability of $E = \{H, TH, TT\}$ is $q + 2r$, and so is the probability of its complement $E^C = \{T, HT, HH\}$. Thus, the probability of E and that of E^C should be 1/2. This holds even if the probability of getting the ball with label 1 out of the urn is not equal to the probability of getting the other ball with label 2. That is to say that the probability of E is 1/2, no matter the values of q and r. Notice that the sample space is {H, T, HH,HT, TH, TT}, so we should have $2q+4r = 1$, thus the probability of E is $p(E) = q+2r = 2q+4r/2 = 1/2$. All we assumed here was that (1) p(**H**) = p(**T**) = q, and (2) p(**HH**) = p(**HT**) = p(**TH**) = =p(**TT**) = r.

Therefore, the probability of our event E is not equal to 1, but its Omega is so: $\Omega_E = 1/2 + 1/4 + 1/4 = 1$. In the literature, the Omega of a set S of finite sequences of 0's and 1's, so-called *binary strings*, is defined as $\Omega_S = \Sigma_{\sigma \in S} 2^{-|\sigma|}$, where $|\sigma|$ denotes the length of σ (ΩS is also called the *weight* of the set S; see, e.g., [2, p. 201]). Let us note that we have identified **H** with 0 and **T** with 1.

Fair Probability Measures

This number Ω_S bears the grand title of probability, for *probably*, two reasons: (I) If S is prefix-free, i.e., no element of S is a proper prefix of another element of S [2, p. 74], then by Kraft's In-equality [2, p. 125], we have $0 \leqslant \Omega_S \leqslant 1$. (II) If S is prefix-free, then Ω_S is equal to the probability that the binary expansion (after zero and dot) of a given real number in the unit interval [0, 1] contains a string from S as a prefix (see [2, p. 4]).

These are mathematically proven facts, and we do not dispute them (see [3, Prop. 2.9, Cor. 3.7]). But another interpretation persists in the literature to the effect that Ω_S is the probability of getting a string in S if we toss a fair coin a finite, but indefinite, number of times (see [1]). Notice that $\tilde{E} = \{0, 10, 11\}$ is prefix-free (and we have $\Omega_{\tilde{E}} = 1$). But how can the probability of getting a string from $\tilde{\mathrm{E}}$ by tossing a fair coin (whose one side is 0 and the other is 1) be equal to $1 (= \Omega_{\tilde{E}})$? Let us compute the probability of getting a string from $\tilde{E}$ by tossing a fair coin an indefinitely finite number of times. Our sample space is $\{0, 1\}^+$, the set of all nonempty binary strings. Let π_1 be the common probability of 0 and 1 (recall the definition of a fair coin). Let π_2 be the common probabilities of 00, 01, 10, and 11. For each $n > 0$, let π_n be the probability of the binary strings with length n. Since there are 2^n such strings, then we should have:

$$\sum_{n=1}^{\infty} 2^n \pi_n = 1.$$

Now, the probability of $\tilde{E}$ is $\pi_1 + 2\pi_2$, which is much less than 1, since:

$$\pi(\tilde{E}) = \pi_1 + 2\pi_2 = 2\pi_1 + 4\pi_2/2 \leqslant \Sigma^{\infty}_{n=1} 2^n \pi_n/2 = \tfrac{1}{2}.$$

This misunderstanding happens for a special prefix-free set: the set of halting programs, that is, the binary codes of the input-free programs that eventually stop after running (and do not loop forever). Call that set **H**. The number Ω in the literature is $\Omega_{\mathbf{H}}$, which equals to $\Sigma_{p\ halts} 2^{-|p|}$, where p ranges over the binary codes of input-free programs (see [1]). If N_l is the number of halting input-free programs with length l, then $\Omega = \Sigma^{\infty}_{l=1} = 1\ N_l 2^{-l}$(see [4, p. 1]). If π_l is the probability of getting any binary string with length l, then, by $\Sigma^{\infty}_{l=1} 2^l \pi_l = 1$, it can be shown that the halting probability measured by π ($\mathrm{HP}\pi = \Sigma^{\infty}_{l=1} = N_l \pi_l$) is less than the celebrated Omega number Ω (see [3, Thm. 3.4]). This holds more generally for every prefix-free set. For a set S of binary strings, let $N_l(S)$ be the number of elements of S with length l. Then the probability of S is, by definition, $\pi(S) = \Sigma^{\infty}_{l=1} = 1\ N_l(S)\pi_l$.

Theorem 1 ($\pi(S) < \Omega_S$)

For every fair probability measure π on $\{0, 1\}^+$, that is a sequence $\{\pi l\}^{\infty}_{l=1}$ of nonnegative real numbers that satisfy $\Sigma^{\infty}_{l=1} 2^l\pi_l = 1$, and every prefix-free set S that contains at least two strings with different lengths, we have $\pi(S) < \Omega_S$.

Proof:

On the one hand, for every l we have $N_l(S)\,\pi_l \leqslant N_l(S)2^{-l}$, and on the other hand, there are at least two l's with $N_l(S) > 0$ and there is at most one l with $\pi_l = 2^{-l}$. Thus, for at least one l we should have $Nl(S)\,\pi_l < N_l(S)2^{-l}$. Therefore, $\pi(S) = \Sigma^{\infty}_{l=1} N_l(S)\,\pi_l < \Sigma^{\infty}_{l=1} N_l(S)2^{-l} = \Omega_S$. ☒

Indeed, Ω_S is the limit of an interesting sequence of probabilities. Fix N to be a positive natural number. Let us toss a fair coin N times and compute the probability of getting an end-extension of a member of S. Our event is, in other words, the set of all binary strings with length N that contain a member of S as a prefix. This makes sense only if S is prefix-free.

Theorem 2 (probability of end-extensions of *S* with fixed length)

For a fixed $N > 0$ and a prefix-free set $S \subseteq \{0, 1\}^+$, the probability that a member of S appears as a prefix of the binary string after N times of tossing a fair coin (with one side 0 and the other 1) is the following number:

$$\sum_{\sigma \in S}^{|\sigma| \leqslant N} 2^{-|\sigma|}.$$

Proof:

If $\{\sigma_i\}^m_{i=1}$ is the set of all string in S with length $\leqslant N$, then our event consists of all $\sigma_i\tau$ where τ is an arbitrary binary string with length $N - |\sigma i|$, and $i = 1, \ldots, m$. Thus, there are $\Sigma^{|\sigma| \leqslant N}_{\sigma \in S} 2^{N-|\sigma|}$ binary strings with length N that have some σ_i as a prefix. Therefore, the probability is:

$1/2N\ \Sigma^{\sigma| \leqslant N}_{\sigma \in S}\ 2^{N-|\sigma|}$, or $\Sigma^{|\sigma| \leqslant N}_{\sigma \in S} 2^{-|\sigma|}$ x☒

Let us notice that:

$\Omega_S = lim \quad \sigma| \leqslant N$
$N \to \infty \quad \Sigma \quad 2^{-|\sigma|}$.
$\sigma \in S$

Unfair Probability Measures

We saw that it is very improbable that an Omega number could be the probability of getting binary strings from given prefix-free sets by tossing a fair coin for finitely many times. We called a probability measure on the binary strings *fair* when strings with equal lengths have equal probabilities. We denoted a fair probability of the strings with length l by π_l and noted that $\Sigma^{\infty}_{l=1} 2^l \pi_l = 1$. When, then, could we have $\pi(S) = \Omega_S$? Theorem 1 says not if S contains two strings with different lengths. So, let S consist of some binary strings with a fixed length ℓ. Then $\pi(S) = N_\ell(S)\pi_\ell$, and $\Omega_S = N_\ell(S)2^{-\ell}$. The equality $\pi(S) = \Omega_S$ holds if and only if $\pi_\ell = 2^{-\ell}$; thus, by $\Sigma^{\infty}_{l=1} 2^l \pi_l = 1$, for every $l \neq \ell$ we must have $\pi_l = 0$. So, for a non-uniform, but still a fair, probability measure μ, defined as $\mu(\sigma) = 2^{-|\sigma|}$ if $|\sigma| = \ell$, and $\mu(\sigma) = 0$ otherwise, we can have $\mu(S) = \Omega_S$, if S consists of some binary strings with a fixed length ℓ only.

The following table summarizes our observations about Ω_S and its approximations for a prefix-free set S (cf. [3, Lem. 3.1 and Cor. 3.7(1)] and Theorem 2):

$|\sigma| \leqslant \ell$

$\Sigma \quad 2^{-|\sigma|}$ = the probability of getting a member of S after tossing a fair coin for ℓ times:

$\sigma \in S$

$|^{\sigma}| \leqslant {}^{N}$

$\Sigma \quad 2^{-|\sigma|}$ = the probability of getting an end-extension of a member of S after tossing a fair coin $\sigma \in S$ for N times

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$\Sigma_{\sigma \in S} \ ^{2-|\sigma|}$ = the probability of getting an end-extension of a member of S after tossing a (= ΩS) fair coin for infinitely many times.

Let us, finally, neglect the fairness of our coin. One can define a probability measure on the binary strings $\{0, 1\}^+$ in such a way that the halting probability measured by it becomes Ω, or any number strictly between 0 and 1. We recall that a probability measure maps each string $\sigma \in \{0, 1\}^+$ to a non-negative real number $\Pi(\sigma)$, such that the equation $\Sigma_{\sigma \in \{0,1\}+} \Pi(\sigma)= 1$ is true. If Π is not fair, then we may have $\Pi(\tau) \neq \Pi(\tau')$ for some $\tau, \tau' \in \{0, 1\}^+$ with equal lengths.

Fix a real number $\alpha \in (0, 1)$. For a binary string σ, define $\Pi\alpha(\sigma)$ as $\alpha / \Omega^{2-|\sigma|}$ if it is the binary code of an input-free halting program, and let $\Pi\alpha(\sigma)$ be $1-\alpha / {}^{2|\sigma|(2|\sigma|-}N_{|\sigma|)}$ otherwise. That is:

$\alpha/^{2|\sigma|} 1/ \Omega$, if $\sigma \in H$

$\Pi\alpha(\sigma) = 1-\alpha/^{2|\sigma|}\ 1/2^{|\sigma|}-N_{|\sigma|}$, if $\sigma \notin H$.

Then the haltingPprobability by $\Pi\alpha$, that is $\Pi\alpha(\mathbf{H})$, becomes:

$\Sigma_{\sigma \in \boldsymbol{H}} \Pi\alpha(\sigma) = \alpha / \Omega\, \Sigma_{\sigma \in \boldsymbol{H}} 1/2^{|\sigma|} = \alpha$

This $\Pi\alpha$ is a probability measure since:

$\Sigma_{\sigma \in \{0,1\}} + \Pi\alpha(\sigma) = \alpha + (1- \alpha)\, \Sigma_{\sigma \notin \mathbf{H}}\ 1/\ ^{2|\sigma|}\ 1/2^{|\sigma|}\,(^{|\sigma|}-N_{|\sigma|}) = \alpha + (1- \alpha)$ $\Sigma^{\infty}{}_{l=1} 1/2^{l} = 1$.

Therefore, by nonstandard probability measures, every real number strictly between zero and one can be a halting probability.

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